Appendix. Proofs

Proof of Proposition 1. From known results (Ahuja, Magnanti, and Orlin 1993, 103-104) the inequalities (4) are feasible, if and only if the constraint graph $\widehat{GS}(S)$ does not have a positive length cycle. By definition, the flat-solution S is implementable if and only if (3) is feasible. Hence it is sufficient to show that the inequalities (3) are equivalent to (4).

Observe that (4) is simply (3) shifted by D: for (3) $\Rightarrow$ (4) simply take $D = D' = 0$. For (3) $\Leftarrow$ (4), let $\{(\hat{t}^k_n), D, D'\}$ be a solution of (4), and let $t^k_n = \hat{t}^k_n - D$. Then clearly (t^k_n) satisfies (3), and the only constraint that does not follow immediately is given by

$$\begin{aligned} \hat{t}^k_{d^k} - D' &\le l^k \\ t^k_{d^k} + D - D' &\le l^k \\ t^k_{d^k} &\le l^k + D' - D, \end{aligned}$$

and $D' - D \le 0$ implies $t^k_{d^k} \le l^k$. □

We next prove the following useful lemma:

Lemma 5. *For any $\alpha \in \mathcal{T}^n$ and $(n,t,t') \in \mathcal{N}_{\mathcal{T}}$, if $\alpha < t'$ then $\alpha \le t$.*

Proof. Let $i, j \in \mathbb{Z}$ be such that α is the ith time point in $\mathcal{T}^n$ and t' is the jth, where $\mathcal{T}^n$ is ordered from smallest to largest time. Then t is the $(j-1)$th time point in $\mathcal{T}^n$ by construction of $\mathcal{N}_{\mathcal{T}}$. Thus $\alpha < t'$ implies $i < j$, and thus $i \le j - 1$. Hence $\alpha \le t$. □

Proof of Lemma 4. Consider the induction hypothesis

$$H(i): \quad t_i \ge e^k + \tau\left(\widehat{P}^{\to(n_i,t_i,t'_i)}\right), \quad \text{for some } i = 1, \ldots, r-1.$$

It is true by assumption for $i = 1$, and since $\tau(\widehat{P}^{\to(n_1,t_1,t'_1)}) = 0$. Suppose that $H(i)$ is true for some $i \in \{1, \ldots, r-1\}$. Recall that $\mathcal{A}^K_{\mathcal{T}} = \bigcup_{k\in K} \mathcal{A}^k_{\mathcal{T}}$, then observe that there exists $k' \in K$ such that $((n_i,t_i,t'_i),(n_{i+1},t_{i+1},t'_{i+1})) \in \mathcal{A}^{k'}_{\mathcal{T}}$, so by $[\![5]\!]$ on $\mathcal{A}^{k'}_{\mathcal{T}}$

$$\begin{aligned} t'_{i+1} &> t_i + \tau_{(n_i,n_{i+1})} \\ &\ge e^k + \tau(\widehat{P}^{\to(n_i,t_i,t'_i)}) + \tau_{(n_i,n_{i+1})} \\ &= e^k + \tau(\widehat{P}^{\to(n_{i+1},t_{i+1},t'_{i+1})}) \end{aligned}$$

By the above lemma, and since $(n_{i+1},\, e^k + \tau(\widehat{P}^{\to(n_{i+1},t_{i+1},t'_{i+1})})) \in \mathcal{T}$ it follows that

$$t_{i+1} \ge e^k + \tau(\widehat{P}^{\to(n_{i+1},t_{i+1},t'_{i+1})}),$$

and $H(i)$ holds. □

Proof of THEOREM 1. First observe that since the flat-solution S is non-implementable and the solution graph $GS(S)$ is acyclic, the path P, defined in the theorem, exists. Specifically, from LEMMA 2 there exists a simple path, P, in $GS(S)$, from $t^{k_1}_{o^{k_1}}$ to $t^{k_2}_{d^{k_2}}$ for some $k_1, k_2 \in K$, such that $e^{k_1} + \rho(P) > l^{k_2}$. Moreover, this path can always be chosen without any holding time, that is, for all t^k_n in P: $E_{t^k_n} = e^{k_1} + \rho(P^{\rightarrow t^k_n})$. To see this, walk up the graph from $t^{k_2}_{d^{k_2}}$ following the $\arg\max_{k \in \kappa} E_{t^k_n}$ commodity at each consolidation.

Suppose that S is representable in $\mathcal{G}^K_{\mathcal{T}'}$, then from Observation 1 there exists the simple path $\widehat{P}$ in $(\mathcal{N}_{\mathcal{T}'}, \mathcal{A}^K_{\mathcal{T}'})$ that corresponds to P, in particular having

$$a = ((n_{|\widehat{P}|-1}, t_{|\widehat{P}|-1}, t'_{|\widehat{P}|-1}), (d^{k_2}, t_{|\widehat{P}|}, t'_{|\widehat{P}|})) \in \mathcal{A}^{k_2}_{\mathcal{T}'},$$

and with

$$\tau(\widehat{P}^{\rightarrow i}) = \rho(P^{\rightarrow t^k_n}),$$

for all $i = (n, t, t')$, the node-interval corresponding with a visit to t^k_n. Observe that the path P may visit the node $n \in N$ with multiple commodities at different times, but a single commodity will visit n at most once. Clearly then i has a unique correspondence to t^k_n, and hence

$$E_{t^k_n} = e^{k_1} + \tau(P^{\rightarrow i}).$$

Thus by LEMMA 4 on $\widehat{P}$,

$$t_{|\widehat{P}|} \geq e^{k_1} + \tau(\widehat{P}) = e^{k_1} + \rho(P) > l^{k_2},$$

but this violates $[\![4]\!]$ on $\mathcal{A}^{k_2}_{\mathcal{T}'}$. Thus $a \notin \mathcal{A}^{k_2}_{\mathcal{T}'}$, and there does not exist path $\widehat{P}$ corresponding to P, hence by Observation 2, S is not representable in $\mathcal{G}^K_{\mathcal{T}'}$. □

Proof of THEOREM 2. Suppose that S is representable in $\mathcal{G}^K_{\mathcal{T}'}$, then by Observation 1 there exists a timed-path in $\mathcal{G}^k_{\mathcal{T}'}$ corresponding to the flat-path Q^k; more specifically, there exist timed-arcs

$$((n_1, t_1, t'_1), (n_2, t_2, t'_2)), \; ((n_2, \hat{t}_2, \hat{t}'_2), (n_3, t_3, t'_3)) \in \mathcal{A}^k_{\mathcal{T}'},$$

with $t_2 \leq \hat{t}_2$. Then using LEMMA 5, with $(n_2, \, e^k + \gamma(o^k, n_1) + \tau_{(n_1, n_2)}) \in \mathcal{T}'$, and $[\![3]\!]$ on $\mathcal{A}^k_{\mathcal{T}'}$ implies

$$e^k + \gamma(o^k, n_1) + \tau_{(n_1, n_2)} \leq t_2,$$

and $[\![4]\!]$ on $\mathcal{A}^k_{\mathcal{T}'}$ requires

$$\hat{t}_2 + \tau_{(n_2, n_3)} + \gamma(n_3, d^k) \leq l^k,$$

but this implies

$$e^k + \gamma(o^k, n_1) + \tau_{(n_1, n_2)} + \tau_{(n_2, n_3)} + \gamma(n_3, d^k) \leq l^k.$$

This contradicts the assumption, so at least one of the above timed-arcs cannot exist. Thus the timed-path cannot exist, and by Observation 2, S is not representable in $\mathcal{G}^K_{\mathcal{T}'}$. □

Proof of THEOREM 3. Suppose that S is representable in $\mathcal{G}^K_{\mathcal{T}'}$, then by Observation 1 there exists timed-paths in $\mathcal{G}^K_{\mathcal{T}'}$ corresponding to Q^{k_1}, and Q^{k_2}; more specifically, there exist timed-arc $a = ((n_1, t_1, t'_1), (n_2, t_2, t'_2))$, such that $a \in \mathcal{A}^{k_1}_{\mathcal{T}'}$, and $a \in \mathcal{A}^{k_2}_{\mathcal{T}'}$. Observe that $[\![4]\!]$ on $\mathcal{A}^{k_2}_{\mathcal{T}'}$ requires

$$t_2 + \gamma(n_2, d^{k_2}) \leq l^{k_2},$$

and $[\![3]\!]$ on $\mathcal{A}^{k_1}_{\mathcal{T}'}$ requires

$$e^{k_1} + \gamma(o^{k_1}, n_1) + \tau_{(n_1,n_2)} < t'_2.$$

By assumption $(n_2,\, e^{k_1} + \gamma(o^{k_1}, n_1) + \tau_{(n_1,n_2)}) \in \mathcal{T}'$, so LEMMA 5 gives

$$e^{k_1} + \gamma(o^{k_1}, n_1) + \tau_{(n_1,n_2)} \leq t_2,$$

but this implies that

$$e^{k_1} + \gamma(o^{k_1}, n_1) + \tau_{(n_1,n_2)} + \gamma(n_2, d^{k_2}) \leq l^{k_2}.$$

Since this contradicts assumptions, then $a \notin \mathcal{A}^{k_1}_{\mathcal{T}'}$, or $a \notin \mathcal{A}^{k_2}_{\mathcal{T}'}$, and hence the timed-path cannot exist, and by Observation 2, S is not a representable solution in $\mathcal{G}^K_{\mathcal{T}'}$. $\square$

Proof of THEOREM 4. Suppose S is representable in $\mathcal{G}^K_{\mathcal{T}'}$, then from Observation 1 there exists the simple path $\widehat{P}$ and simple cycle $\mathring{\widehat{P}}$ in $(\mathcal{N}_{\mathcal{T}'}, \mathcal{A}^K_{\mathcal{T}'})$ that corresponds to P and $\mathring{P}$ respectively. Specifically, there exists arcs

$$((n_{j-1}, t_{j-1}, t'_{j-1}),\, (n'_j, t_j, t'_j)) = \widehat{P}_{(j)} \in \mathcal{A}^{k_1}_{\mathcal{T}'}, \text{ for } j = 1, \ldots, |\widehat{P}|, \text{ and}$$

$$((n_{|\widehat{P}|+j'}, t_{|\widehat{P}|+j'}, t'_{|\widehat{P}|+j'}), (n'_{|\widehat{P}|+j'+1}, t_{|\widehat{P}|+j'+1}, t'_{|\widehat{P}|+j'+1})) = \mathring{\widehat{P}}_{(j'+1)} \in \mathcal{A}^K_{\mathcal{T}'} \text{ for } j' = 0, \ldots, |\mathring{\widehat{P}}|,$$

such that

$$(n_{|\widehat{P}|}, t_{|\widehat{P}|}, t'_{|\widehat{P}|}) = (n'_{|\widehat{P}|+|\mathring{\widehat{P}}|+1}, t_{|\widehat{P}|+|\mathring{\widehat{P}}|+1}, t'_{|\widehat{P}|+|\mathring{\widehat{P}}|+1}), \text{ and } \mathring{\widehat{P}}_{(1)} \in \mathcal{A}^{k_1}_{\mathcal{T}'}.$$

Observe that $\tau(\widehat{P}^{\to i}) = \rho(P^{\to t^k_n})$ for all $i = (n, t, t')$, that is, the node-interval in $\widehat{P}$ uniquely corresponding with a visit to t^k_n, so by LEMMA 4 on $\widehat{P}$:

$$t_{|\widehat{P}|} \geq e^{k_1} + \tau(\widehat{P}) = e^{k_1} + \rho(P).$$

Assume for a moment that $\mathring{\widehat{P}}$ travels around the corresponding cycle $\mathring{P}$ as early as possible, that is, without waiting at a node longer than necessary. Let $r = |\mathring{P}|$, let $\eta_1, \eta_r \in N$ be the nodes visited by $\mathring{P}_{(1)}$ and $\mathring{P}_{(r)}$ respectively, and let i^0_r, $i^1_1 \in \mathcal{N}_{\mathcal{T}'}$ be the node-interval corresponding to the first visit to η_r, and the first return to η_1 in $\mathring{\widehat{P}}$ respectively. Observe that joining P and $\mathring{P}^{\to \mathring{P}_{(r)}}$ yields a simple path, and so by LEMMA 4 on $\widehat{P} \cup \mathring{\widehat{P}}^{\to i^0_r}$:

$$\begin{aligned} t_{|\widehat{P}|+|\mathring{\widehat{P}}^{\to i^0_r}|} &\geq e^{k_1} + \tau(\widehat{P}) + \tau(\mathring{\widehat{P}}^{\to i^0_r}) \\ &= e^{k_1} + \rho(P) + \rho(\mathring{P}^{\to \mathring{P}_{(r)}}). \end{aligned}$$

Following from the proof in LEMMA 4, there exists $k' \in K$ such that

$$((\eta_r, t_{s^1-1}, t'_{s^1-1}), (\eta_1, t_{s^1}, t'_{s^1})) \in \mathcal{A}^{k'}_{\mathcal{T}'} \cap \mathring{\widehat{P}},$$

with $s^1 = |\widehat{P}| + |\mathring{\widehat{P}}^{\to i_1^1}|$. By $[\![5]\!]$ on $\mathcal{A}^{k'}_{\mathcal{T}'}$

$$\begin{aligned} t'_{s^1} &> t_{s^1-1} + \tau_{(\eta_r,\eta_1)} \\ &\geq e^{k_1} + \rho(P) + \rho(\mathring{P}^{\to \mathring{P}_{(r)}}) + \tau_{(\eta_r,\eta_1)} \\ &= e^{k_1} + \rho(P) + \rho(\mathring{P}). \end{aligned}$$

By Lemma 5, and since $(\eta_1,\, e^{k_1} + \rho(P) + \rho(\mathring{P})) \in \mathcal{T}'$ it follows that

$$t_{s^1} \geq e^{k_1} + \rho(P) + \rho(\mathring{P}).$$

Thus the first return to η_1, $\widehat{P} \cup \mathring{\widehat{P}}^{\to i_1^1}$, is a simple path in $(\mathcal{N}_{\mathcal{T}'}, \mathcal{A}^K_{\mathcal{T}'})$, i.e., not a cycle. Intuitively then, for $m \leq \mathring{m}$ walks around the $\mathring{P}$ cycle, the corresponding $\widehat{P} \cup \mathring{\widehat{P}}^{\to i_1^m}$ is also a simple path in $(\mathcal{N}_{\mathcal{T}'}, \mathcal{A}^K_{\mathcal{T}'})$ with

$$\begin{aligned} t_{s^m} &\geq e^{k_1} + \tau(\widehat{P}) + \tau(\mathring{\widehat{P}}^{\to i_1^m}) \\ &= e^{k_1} + \rho(P) + m\,\rho(\mathring{P}), \end{aligned}$$

and $s^m = |\widehat{P}| + |\mathring{\widehat{P}}^{\to i_1^m}|$. Consider now that $\mathring{\widehat{P}}$ can delay its walk around the corresponding $\mathring{P}$ cycle by waiting, say, at time point $(\eta, t) \in \mathcal{T}'$ with $t \geq t_{|\widehat{P}|}$. By definition of $\mathcal{T}'$, each delay is in increments of at most length $\rho(\mathring{P})$ when $t \leq e^{k_1} + \rho(P) + (\mathring{m} - 1)\rho(\mathring{P})$, and a dispatch from η at time $t + \rho(\mathring{P})$, will arrive to the subsequent node η' at time $t + \rho(\mathring{P}) + \tau_{(\eta,\eta')}$, i.e., this time spent waiting is not lost. Delays of a size smaller than $\rho(\mathring{P})$ indicate a more refined discretization and can be ignored. Thus we can interpret the $m \leq \mathring{m}$ above as either walks around the $\mathring{P}$ cycle, or delays at a node. In either case, the choice of $\mathring{m}$ means that $\widehat{P} \cup \mathring{\widehat{P}}^{\to i_1^{\mathring{m}}}$ is a simple path in $(\mathcal{N}_{\mathcal{T}'}, \mathcal{A}^K_{\mathcal{T}'})$, and, for $\mathring{\widehat{P}}$ to complete its cycle, it first requires a dispatch from $n_{|\widehat{P}|}$ after $t_{s^{\mathring{m}}}$. It should be clear from $[\![5]\!]$ and Lemma 5 that $t_{|\widehat{P}|+|\mathring{\widehat{P}}|+1} \geq t_{s^{\mathring{m}}}$. But $t_{s^{\mathring{m}}} > l^{k_1}$ violates $[\![4]\!]$ on $\mathcal{A}^{k_1}_{\mathcal{T}'}$, thus $\mathring{\widehat{P}}_{(1)} \notin \mathcal{A}^{k_1}_{\mathcal{T}'}$, and so there exists no cycle $\mathring{\widehat{P}}$ corresponding to $\mathring{P}$. Hence by Observation 2, S is not representable in $\mathcal{G}^K_{\mathcal{T}'}$. $\square$

References

Ahuja RK, Magnanti TL, Orlin JB, 1993 *Network Flows - Theory, Algorithms and Applications* (Englewood Cliffs, NJ: Prentice-Hall).