A. Reduction from the multiple destination case [Online]

In this section, we show how our model can accommodate multiple destinations for different orders. We then observe that the same reasoning can be applied to multiple sources, but cannot be extended to the combination of the two, i.e., the case of multiple sources and destinations.

Given a coalition $S \subseteq N$, let $t_1^S, \dots, t_d^S$ be the destinations and $k_{t_1}^S, \dots, k_{t_d}^S$ be the total demands for each destination. We introduce a super sink t^* and arcs $r_j = (t_j, t^*)$ for all $j = 1, \dots, d$, with per unit cost $c_{r_j} = 0$ and capacity $u_{r_j} = k_{t_j}^S$. Finally, we set total demand at node t^* equal to $\sum_{j=1}^d k_{t_j}^S$.

This transformation of the graph leads to an equivalent formulation of the minimum cost flow problem for coalition S, where the single destination t^* is used.

A similar reduction can be applied to the case of multiple sources. The case of multiple sources and destinations, however, cannot be modeled with our formulation, as it requires the definition of a multicommodity flow problem in order to distinguish the path followed by each commodity, which might not be conserved otherwise (Ahuja, Magnanti, and Orlin 1993).

B. Proof of Proposition 1 [Online]

Proposition 1. *The parametric minimum cost flow game $c(\lambda)$ is subadditive and has a non-empty core for all $\lambda \in \Lambda$.*

Proof. In what follows, we assume that $\lambda \in \Lambda$ is fixed and consider the parametric minimum cost flow game $c(\lambda)$ for this given value. To simplify notation, we write c_S for $c_S(\lambda)$ for all $S \subseteq N$, and prove that the core of the minimum cost flow game (N, c) is non-empty.

We follow Owen (1975) to prove this claim by showing that the minimum cost flow game without integrality constraints is a balanced cost sharing game. Balancedness of the game is equivalent to non-emptiness of the core. Integral optimal solutions are found by solving the linear relaxation P_L^S of problem P^S obtained by substituting the flow integrality constraint $f_r^S \in \mathbb{N}_{\geq 0}$ with $f_r^S \in \mathbb{R}_{\geq 0}$. This allows for our proof while guaranteeing the same result for our case as integral optimal solutions can be found by solving the relaxed problem. We assume that problem P^S if feasible for all coalitions $S \subseteq N$, so P_L^S is also feasible.

To define balancedness of a game, the concept of a balanced map is required. A map $\gamma : 2^N \setminus \{\emptyset\} \to [0, +\infty)$ is called *balanced* for N if

$$\sum_{S \in 2^N \setminus \{\emptyset\}} \gamma(S) \underline{e}^S = \underline{e}^N \tag{1}$$

where, for each $\emptyset \neq S \subseteq N$, the vector $\underline{e}^S \in \mathbb{R}^{|N|}$ is such that

$$e_i^S := \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall i \in N.$$

A cost sharing game (N,c) is balanced if, for every balanced map γ for N, we have $\sum_{S\in 2^N\setminus\{\emptyset\}}\gamma(S)c_S \geq c_N$.

In what follows, we prove that the minimum cost flow game is balanced. Let γ be a balanced map for N. Then we have

$$\begin{aligned}\sum_{S\in 2^N\setminus\{\emptyset\}}\gamma(S)k^S &= \sum_{S\in 2^N\setminus\{\emptyset\}}\sum_{i\in S}\gamma(S)k_i\\ &=\sum_{i\in N}\left[\sum_{S\in 2^N\setminus\{\emptyset\}:i\in S}\gamma(S)\right]k_i\\ &=\sum_{i\in N}k_i\\ &=k^N\end{aligned}$$

Now, we have $c_S=\sum_{r\in R^S}c_r f_r^S$ by definition of the cost game, where the optimal flow allocation $\{f_r^S\}_{r\in R^S}$ satisfies constraints (3b) (3c), (3d),(3e). Then,

$$\begin{aligned}\sum_{S\in 2^N\setminus\{\emptyset\}}\gamma(S)c_S &= \sum_{S\in 2^N\setminus\{\emptyset\}}\sum_{r\in R^S}\gamma(S)c_r f_r^S\\ &=\sum_{r\in R^N}c_r\sum_{S\in 2^N\setminus\{\emptyset\}:r\in R^S}\gamma(S)f_r^S\\ &=\sum_{r\in R^N}c_r\hat{f}_r\end{aligned}\tag{2}$$

where $\hat{f}_r := \sum_{S\in 2^N\setminus\{\emptyset\}:r\in R^S}\gamma(S)f_r^S$ for all arcs $r\in R^N$. It follows that the vector $\{\hat{f}\}_{r\in R^N}$ is feasible for the relaxed problem P_L^N of the grand-coalition. Indeed, for all $v\in V^N\setminus\{s,t\}$

$$\begin{aligned}\sum_{r\in\delta_N^-(v)}\hat{f}_r &= \sum_{r\in\delta_N^-(v)}\sum_{S\in 2^N\setminus\{\emptyset\}:r\in R^S}\gamma(S)f_r^S\\ &=\sum_{S\in 2^N\setminus\{\emptyset\}}\gamma(S)\sum_{r\in\delta_S^-(v)}f_r^S\\ &=\sum_{S\in 2^N\setminus\{\emptyset\}}\gamma(S)\sum_{r\in\delta_S^+(v)}f_r^S\\ &=\sum_{r\in\delta_N^+(v)}\sum_{S\in 2^N\setminus\{\emptyset\}:r\in R^S}\gamma(S)f_r^S\\ &=\sum_{r\in\delta_N^+(v)}\hat{f}_r.\end{aligned}$$

Similarly, we have

$$\sum_{r\in\delta_N^+(s)}\hat{f}_r = \sum_{r\in\delta_N^+(s)}\sum_{S\in 2^N\setminus\{\emptyset\}:r\in R^S}\gamma(S)f_r^S$$

$$\begin{aligned}
&= \sum_{S \in 2^N \setminus \{\emptyset\}} \gamma(S) \sum_{r \in \delta_S^+(s)} f_r^S \\
&= \sum_{S \in 2^N \setminus \{\emptyset\}} \gamma(S) k^S \\
&= k^N.
\end{aligned}$$

The same holds for constraint (3e).

Finally, $\hat{f}_r \geq 0$ for all $r \in R^N$ because of the non-negativity of the coefficients $\gamma(S)$.

As $\{\hat{f}\}_{r \in R^N}$ is feasible for P_L^N, it follows that

$$c_N \leq \sum_{r \in R^N} c_r \hat{f}_r. \tag{3}$$

Then, we see from (2) and (3) that $\sum_{S \in 2^N \setminus \{\emptyset\}} c_S \geq c_N$, so (N, c) is balanced. $\square$

As the proof is independent of λ, this result holds for all $\lambda \in \Lambda$.

C. Proof of Proposition 2 [Online]

Proposition 2. *If there exists* $\hat{\lambda} \in \Lambda$ *and* $\hat{S} \subset N$ *such that* $\Phi_{\hat{S}}(\hat{\lambda}) > c_{\hat{S}}(\hat{\lambda})$, *then*

$$\forall \varepsilon \in \left(0, \frac{\Phi_{\hat{S}}(\hat{\lambda}) - c_{\hat{S}}(\hat{\lambda})}{2K}\right): \quad \Phi(\lambda) \notin \mathfrak{C}(\lambda) \quad \forall \lambda \in (\hat{\lambda} - \varepsilon, \hat{\lambda} + \varepsilon), \tag{4}$$

where $K \geq 0$ *is a Lipschitz constant of the functions* $\{c_S(\lambda)\}_{S \subseteq N}$ *and* $\{\Phi_S(\lambda)\}_{S \subseteq N}$, *i.e,* $\forall \lambda', \lambda'' \in \Lambda$, $|c_S(\lambda') - c_S(\lambda'')| \leq K \cdot |\lambda' - \lambda''|$ *and* $|\Phi_S(\lambda') - \Phi_S(\lambda'')| \leq K \cdot |\lambda' - \lambda''|$ *for all* $S \subseteq N$.[1]

Proof. From the hypothesis, it follows that $\Phi(\hat{\lambda}) \notin \mathfrak{C}(\hat{\lambda})$. Because of piecewise linearity of the cost curves $c_S(\lambda)$ and the marginal cost functions $\Phi_S(\lambda)$ ($S \subseteq N$), these functions satisfy the Lipschitz property, i.e., there exists a constant $K \geq 0$ such that $|c_S(\lambda') - c_S(\lambda'')| \leq K \cdot |\lambda' - \lambda'|$ for all $\lambda', \lambda'' \in \Lambda$ and for all $S \subseteq N$, and similarly for the marginal cost functions $\Phi_S(\lambda)$ (because we deal with a finite number of functions, we can use the same Lipschitz constant for all functions by taking the maximum of all of their separate Lipschitz constants). In particular, we obtain that $\Phi_{\hat{S}}(\lambda)$ is bounded from below by $-K \cdot |\hat{\lambda} - \lambda| + \Phi_{\hat{S}}(\hat{\lambda})$ for all $\lambda \in \Lambda$, while $c_{\hat{S}}(\lambda)$ is bounded from above by $K \cdot |\hat{\lambda} - \lambda| + c_{\hat{S}}(\hat{\lambda})$ for all $\lambda \in \Lambda$. These bounds are obtained directly from the Lipschitz property, as shown here for the case of $c_{\hat{S}}(\lambda)$:

$$\begin{aligned}
&|c_{\hat{S}}(\hat{\lambda}) - c_{\hat{S}}(\lambda)| \leq K \cdot |\hat{\lambda} - \lambda| \\
\Leftrightarrow\ &|c_{\hat{S}}(\lambda) - c_{\hat{S}}(\hat{\lambda})| \leq K \cdot |\hat{\lambda} - \lambda| \\
\Leftrightarrow\ &+K \cdot |\hat{\lambda} - \lambda| \geq c_{\hat{S}}(\lambda) - c_{\hat{S}}(\hat{\lambda}) \geq -K \cdot |\hat{\lambda} - \lambda| \\
\Rightarrow\ &c_{\hat{S}}(\lambda) - c_{\hat{S}}(\hat{\lambda}) \leq K \cdot |\hat{\lambda} - \lambda| \\
\Leftrightarrow\ &c_{\hat{S}}(\lambda) \leq K \cdot |\hat{\lambda} - \lambda| + c_{\hat{S}}(\hat{\lambda})
\end{aligned}$$

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[1] In this case, the constant K can be chosen as the highest slope of all functions.

Because of these two bounds for $\Phi_{\hat{S}}(\lambda)$ and $c_{\hat{S}}(\lambda)$, the difference $\Phi_{\hat{S}}(\lambda) - c_{\hat{S}}(\lambda)$ satisfies $\Phi_{\hat{S}}(\lambda) - c_{\hat{S}}(\lambda) \geq -2K|\hat{\lambda} - \lambda| + \Phi_{\hat{S}}(\hat{\lambda}) - c_{\hat{S}}(\hat{\lambda})$. Hence, for any $\varepsilon \in \left(0, \frac{\Phi_{\hat{S}}(\hat{\lambda}) - c_{\hat{S}}(\hat{\lambda})}{2K}\right)$ and any $\lambda \in (\hat{\lambda} - \varepsilon, \hat{\lambda} + \varepsilon)$, we obtain $\Phi_{\hat{S}}(\lambda) - c_{\hat{S}}(\lambda) > 0$, which implies that $\Phi(\lambda) \notin \mathfrak{C}(\lambda)$ and concludes the proof. $\square$

D. Proof of Theorem 1 [Online]

THEOREM 1. *In the case of cooperation on a corridor with identical players, whether the Shapley value is in the core or not depends on the value of the ratio* $\frac{k}{u}$ *compared to the size* $n = |N| \geq 2$ *of the grand coalition as follows:*

$$\Phi(0) \in \mathfrak{C}(0) \qquad \textit{for all} \qquad 0 \leq \frac{k}{u} \leq 1, \tag{5a}$$

$$\Phi(\lambda) \in \mathfrak{C}(\lambda) \quad \forall \lambda > 0 \qquad \textit{if and only if} \qquad \frac{(2n-1)(n-2)}{2n^2 - 3n} \leq \frac{k}{u} \leq 1 \tag{5b}$$

$$\Phi(\lambda) \in \mathfrak{C}(\lambda) \quad \forall {-c_0} \leq \lambda < 0 \qquad \textit{if and only if} \qquad 0 \leq \frac{k}{u} \leq \frac{2n-2}{2n^2 - 3n}. \tag{5c}$$

Proof. Testing stability of the Shapley value means computing the combinations of the parameters n, c_0, λ, u, k for which all the inequalities $\sum_{i \in S} \Phi_i(\lambda) \leq c_S(\lambda)$ (for $S \subseteq N$) are satisfied. To tackle this problem, we will first decompose the minimum cost flow game into two the sum of two games and exploit linearity of the Shapley value to simplify the computations. At that point, we will distinguish three cases based on the value of λ: $\lambda = 0$ is treated first as instrumental to solve the following two cases of $\lambda > 0$ and $\lambda < 0$. For each of the three cases, we will need to discuss several subcases that are required to make the expression of the Shapley value and the cost function explicit. More precisely, we will compute the cost function of each game, the marginal costs, and the Shapley value. From their expressions, the conditions defining the subcases will become clear. In each subcase, we then make the expressions required to solve the system of inequalities $\{\sum_{i \in S} \Phi_i(\lambda) \leq c_S(\lambda) : S \subseteq N\}$ explicit for the parameters mentioned above. At this stage, the inequalities can be solved exactly, which leads to the conditions in (5).

Before delving into the proof, we recall linearity of the Shapley value over the set of cooperative games in characteristic function form (Shapley 1953). Given two games $c^1 = (c^1_S)_{S \subseteq N} \in \mathbb{R}^{2n}$ and $c^2 = (c^2_S)_{S \subseteq N} \in \mathbb{R}^{2n}$ on the same set of players N (where $|N| = n$), the sum $c^1 + c^2$ of the two games is the game defined by $(c^1 + c^2)_S = c^1_S + c^2_S$ for each coalition $S \subseteq N$. Similarly, for a given scalar $\alpha \in \mathbb{R}$, the game αc^1 is defined by $(\alpha c^1)_S = \alpha c_S$ for each coalition $S \subseteq N$. Linearity of the Shapley value implies that $\Phi(c^1 + \alpha c^2) = \Phi(c^1) + \alpha \Phi(c^2)$ (Shapley 1953). As a final remark, we write c for c_0 in the remainder of the proof to simplify the notation.

Case 0: Special cases. We start by dealing with two special cases.

First, we note that the Shapley value is stable in the cases where $k = 0$ or $k = u$, independently of the value of λ or the size n of the grand-coalition. Indeed, if $k = 0$, no orders are transported and all costs are zero, which results in a Shapley value allocation $\Phi(\lambda) \equiv 0$ that is stable (see Inequalities 7). For $k = u$, instead, we observe that the cost of each coalition $S \subseteq N$ satisfies $c_S(\lambda) = \sum_{i\in S} c_i(\lambda)$ because all arcs are saturated. Thus, individual rationality of the Shapley value (i.e., $\Phi_i(\lambda) \le c_i(\lambda)$ for all players $i \in N$) implies coalitional rationality and, therefore, stability of the Shapley value itself ($\sum_{i\in S} \Phi_i(\lambda) \le \sum_{i\in S} c_i(\lambda) = c_S(\lambda)$).

Second, in case of a two-players cooperation (i.e., $n = 2$) the Shapley value is stable independently of the values of all other parameters. Indeed, coalitional rationality is implied by individual rationality and efficiency of the Shapley value as only the cases of $|S| = 1$ and $S = N$ remain from the inequalities in (7).

Having taken care of Case 0, we may assume for the rest of the proof that k and u are such that $0 < k < u$, and that $n \ge 3$.

Case 1 1: $\boldsymbol{\lambda = 0}$. For $\lambda = 0$, all players have identical costs and the cost of each coalition $S \subseteq N$ equals $c_S(0) = |S|kc$. We define the game c^0 having cost $c^0_S := |S|kc$ for each coalition $S \subseteq N$. We denote the Shapley value allocation computed for the game c^0 by $\Phi_i(c^0)$. Note that $\Phi_i(0) = \Phi_i(c^0)$. Since all players are symmetric in this game (and, thus, $\Phi_i(0) = \Phi_j(0)$ for all i, j) and the Shapley value is an efficient solution concept (i.e., $\sum_{i\in N} \Phi_i = c^0_N$), the Shapley value allocation $\Phi_i(0)$ to each player $i \in N$ is $\Phi_i(c^0) = kc$.

In this case, it can be observed that $\sum_{i\in S} \Phi_i(c^0) = |S|kc = c^0_S$ for each coalition $S \subseteq N$. Thus, for all values of k and u such that $0 < \frac{k}{u} < 1$, we have $\Phi(c^0) \in \mathfrak{C}(c^0)$ and coalitional rationality is satisfied in this case.

Case 2: $\boldsymbol{\lambda > 0}$. For a given coalition $S \subseteq N$, the cost function $c_S(\lambda)$ is given by:

$$c_S(\lambda) = \begin{cases} |S|kc & i_p \notin S \quad (6a) \\ |S|kc & i_p \in S \text{ and } |S| \ge \frac{u}{u-k} \quad (6b) \\ (|S|-1)uc + [u - |S|(u-k)](c+\lambda) & i_p \in S \text{ and } |S| < \frac{u}{u-k} \quad (6c) \end{cases}$$

Cases (6a) and (6b) follow from the fact that $|S|k$ orders can be executed without sending flow on arc r_{i_p}. Note that the inequality $|S| \ge \frac{u}{u-k}$ is equivalent to $|S|k \le (|S|-1)u$, meaning that the total amount of orders is less than the total capacity of all players in S different from i_p. The expression in (6c), instead, follows from the fact that arc r_{i_p} has to be used in spite of its higher cost resulting from the positive value of λ (note that the condition $|S| < \frac{u}{u-k}$ is equivalent to $|S|k > (|S|-1)u$), thus $(|S|-1)$ arcs are used fully at a unit cost of c, and the remaining $|S|k - (|S|-1)u$ orders will use arc r_{i_p} at a unit cost $c + \lambda$.

Now, we decompose the minimum cost flow game $c(\lambda)$ into the sum $c(\lambda) = c^0 + \lambda c^+$ of the two games c^0 and c^+, where c^0 has been defined in Case 1, and c^+ is defined as follows:

$$c_S^+ := \begin{cases} 0 & i_p \notin S \ , & \text{(7a)} \\ 0 & i_p \in S \text{ and } |S| \geq \frac{u}{u-k} \ , & \text{(7b)} \\ u - |S|(u-k) & i_p \in S \text{ and } |S| < \frac{u}{u-k} \ . & \text{(7c)} \end{cases}$$

By comparing each condition in (6) and (7), it can be checked that $c_S(\lambda) = c_S^0 + \lambda c_S^+$ for all $S \subseteq N$. Intuitively, the cost game c^+ counts only the amount of orders transported on arc r_{i_p}, so that λc^+ is the additional cost generated by transporting on arc r_{i_p}.

From this decomposition and linearity of the Shapley value, it follows that $\Phi_i(\lambda) = \Phi_i(c^0) + \lambda \Phi_i(c^+)$ for all players $i \in N$, and that, for each coalition $S \subseteq N$, the inequality $\sum_{i \in S} \Phi_i(\lambda) \leq c_S(\lambda)$ can be written as:

$$\sum_{i \in S} \Phi_i(\lambda) \leq c(\lambda) \tag{8a}$$

$$\Leftrightarrow \quad \sum_{i \in S} \big(\Phi_i(c^0) + \lambda \Phi_i(c^+)\big) \leq c_S^0 + \lambda c_S^+ \tag{8b}$$

$$\Leftrightarrow \quad \sum_{i \in S} \Phi_i(c^0) + \lambda \sum_{i \in S} \Phi_i(c^+) \leq c_S^0 + \lambda c_S^+ \tag{8c}$$

$$\Leftrightarrow \quad \lambda \sum_{i \in S} \Phi_i(c^+) \leq \lambda c_S^+ \qquad \Big(\text{as } \sum_{i \in S} \Phi_i(c^0) = c_S^0\Big) \tag{8d}$$

$$\Leftrightarrow \quad \sum_{i \in S} \Phi_i(c^+) \leq c_S^+ \qquad \big(\text{as } \lambda > 0\big) \tag{8e}$$

Therefore, the Shapley value is stable in the game $c(\lambda)$ for $\lambda > 0$ if and only if the Shapley value is stable in the game c^+. Moreover, note that, in (8), we have obtained both independence of λ and a much simpler game to analyze.

In order to compute the expression of the Shapley value, we compute the marginal cost $c_{S \cup \{i\}}^+ - c_S^+$ for each player $i \in N$ and each coalition $S \subseteq N \setminus \{i\}$. We distinguish two cases, namely, whether $i \neq i_p$ or $i = i_p$.

For $i \neq i_p$, we have:

$$c_{S \cup \{i\}}^+ - c_S^+ = \begin{cases} 0 & i_p \notin S & \text{(9a)} \\ k - u & i_p \in S \text{ and } |S| < \frac{k}{u-k} & \text{(9b)} \\ |S|(u-k) - u & i_p \in S \text{ and } \frac{k}{u-k} \leq |S| < \frac{u}{u-k} & \text{(9c)} \\ 0 & i_p \in S \text{ and } \frac{u}{u-k} \leq |S| & \text{(9d)} \end{cases}$$

For $i = i_p$, we know that $i_p \notin S$, and the marginal cost equals

$$c^+_{S\cup\{i_p\}} - c^+_S = \begin{cases} k & |S| = 0 & \text{(10a)} \\ k - |S|(u-k) & \text{and } 0 < |S| < \frac{k}{u-k} & \text{(10b)} \\ 0 & \frac{k}{u-k} \le |S| & \text{(10c)} \end{cases}$$

With those expressions for the marginal costs, we compute the Shapley value allocation to each player. We first treat the case of players $i \neq i_p$ and then the case of player $i = i_p$. We denote the Shapley value allocation to a player $i \neq i_p$ by $\Phi_{-i_p}(c^+)$, which can be computed as follows:

$$\Phi_{-i_p}(c^+) = \frac{1}{n} \sum_{S\subseteq N\setminus\{i\}} \binom{n-1}{|S|}^{-1} (c^+_{S\cup\{i\}} - c^+_S) \tag{11a}$$

$$= \frac{1}{n}\left(\sum_{\substack{S\subseteq N\setminus\{i\}:\\ i_p\in S,\\ \frac{k}{u-k}\le|S|<\frac{u}{u-k}}} \binom{n-1}{|S|}^{-1} (c^+_{S\cup\{i\}} - c^+_S) + \sum_{\substack{S\subseteq N\setminus\{i\}:\\ i_p\in S,\\ |S|<\frac{k}{u-k}}} \binom{n-1}{|S|}^{-1} (c^+_{S\cup\{i\}} - c^+_S) \right) \tag{11b}$$

$$= \frac{1}{n}\left(\sum_{\substack{l=1,\dots,n-1:\\ \frac{k}{u-k}\le l<\frac{u}{u-k}}} \sum_{\substack{S\subseteq N\setminus\{i\}:\\ i_p\in S,\\ |S|=l}} \binom{n-1}{l}^{-1} \big(l(u-k)-u\big) + \sum_{\substack{l=1,\dots,n-1:\\ l<\frac{k}{u-k}}} \sum_{\substack{S\subseteq N\setminus\{i\}:\\ i_p\in S,\\ |S|=l}} \binom{n-1}{l}^{-1} (k-u) \right) \tag{11c}$$

$$= \frac{1}{n}\left(\sum_{\substack{l=1,\dots,n-1:\\ \frac{k}{u-k}\le l<\frac{u}{u-k}}} \binom{n-1}{l}^{-1} (l(u-k)-u)\binom{n-2}{l-1} + \sum_{\substack{l=1,\dots,n-1:\\ l<\frac{k}{u-k}}} \binom{n-1}{l}^{-1} (k-u)\binom{n-2}{l-1} \right) \tag{11d}$$

$$= \frac{1}{n}\left(\sum_{\substack{l=1,\dots,n-1:\\ \frac{k}{u-k}\le l<\frac{u}{u-k}}} \frac{l}{n-1}(l(u-k)-u) + \sum_{\substack{l=1,\dots,n-1:\\ l<\frac{k}{u-k}}} \frac{l}{n-1}(k-u) \right) \tag{11e}$$

We now compute the Shapley value allocation to player $i = i_p$:

$$\Phi_{i_p}(c^+) = \frac{1}{n} \sum_{S\subseteq N\setminus\{i\}} \binom{n-1}{|S|}^{-1} (c^+_{S\cup\{i\}} - c^+_S) \tag{12a}$$

$$= \frac{1}{n}\left(\sum_{\substack{S\subseteq N\setminus\{i_p\}:\\ |S|<\frac{k}{u-k}}} \binom{n-1}{|S|}^{-1} (c^+_{S\cup\{i_p\}} - c^+_S) \right) \tag{12b}$$

$$= \frac{1}{n}\left(\sum_{\substack{l=0,\dots,n-1:\\ l<\frac{k}{u-k}}} \sum_{\substack{S\subseteq N\setminus\{i_p\}:\\ |S|=l}} \binom{n-1}{l}^{-1} \big(k - l(u-k)\big) \right) \tag{12c}$$

$$= \frac{1}{n}\left(\sum_{\substack{l=0,\dots,n-1:\\ l<\frac{k}{u-k}}} \binom{n-1}{l}^{-1} (k - l(u-k)) \binom{n-1}{l} \right) \tag{12d}$$

$$= \frac{1}{n}\left(\sum_{\substack{l=0,\dots,n-1:\\ l<\frac{k}{u-k}}} (k - l(u-k)) \right) \tag{12e}$$

$$= \frac{1}{n}\left(k + \sum_{\substack{l=1,\dots,n-1:\\ l<\frac{k}{u-k}}} (k - l(u-k)) \right) \tag{12f}$$

It is possible to rewrite the expressions in (11e) and (12f) further by discussing the index sets of the summations as, depending on the value of $\frac{k}{u}$, they can be empty. We distinguish three cases that are presented in Table 1 and discuss them now.

In case $0 < \frac{k}{u} \le \frac{1}{2}$, we have $\frac{k}{u-k} \le 1$ and $\frac{u}{u-k} \le 2$, and the index set $\{l = 1,\dots,n-1 : \frac{k}{u-k} \le l < \frac{u}{u-k}\}$ reduces to the single value $l = 1$, while the index set $\{l = 1,\dots,n-1 : l < \frac{k}{u-k}\}$ is empty and the summations with this index set are equal to zero.

In case $\frac{1}{2} < \frac{k}{u} \le \frac{n-1}{n}$, we have $\frac{k}{u-k} \le n-1$ and the index set $\{l = 1,\dots,n-1 : \frac{k}{u-k} \le l < \frac{u}{u-k}\}$ reduces to the singleton $\{l = \left\lceil \frac{k}{u-k} \right\rceil\}$, as $\left\lceil \frac{k}{u-k} \right\rceil$ is the only integer number in $\left[\frac{k}{u-k}, \frac{u}{u-k}\right)$. The index set $\{l = 1,\dots,n-1 : l < \frac{k}{u-k}\}$ equals $\{l = 1,\dots,\left\lceil \frac{k}{u-k} \right\rceil - 1\}$. Indeed, $\left\lceil \frac{k}{u-k} \right\rceil - 1$ is the largest integer number lower than $\frac{k}{u-k}$ (note that $\left\lceil \frac{k}{u-k} \right\rceil - 1 = \left\lfloor \frac{k}{u-k} \right\rfloor$, we opt for this expression for later convenience).

In case $\frac{n-1}{n} < \frac{k}{u} \le 1$, we have $\frac{k}{u-k} > n-1$ and the index set $\{l = 1,\dots,n-1 : \frac{k}{u-k} \le l < \frac{u}{u-k}\}$ is empty, while the index set $\{l = 1,\dots,n-1 : l < \frac{k}{u-k}\}$ equals $\{l = 1,\dots,n-1\}$.

For each subcase that we just defined, we now solve the following set of inequalities: $\sum_{i\in S} \Phi_i(c^+) \le c^+_S$ for all $S \subseteq N$ such that $2 \le |S| \le n-1$. Note that, because the Shapley value is an individually rational and efficient solution concept, the inequalities for $|S| = 1$ and $|S| = n$

Case	$\Phi_{-i_p}(c^+)$	$\Phi_{i_p}(c^+)$
$\boldsymbol{0 < \frac{k}{u} \le \frac{1}{2}}$	$\frac{1}{n}\big(\sum_{l=1} \frac{l}{n-1}(l(u-k)-u)+0\big)$	$\frac{1}{n}(k+0)$
$\boldsymbol{\frac{1}{2} < \frac{k}{u} \le \frac{n-1}{n}}$	$\frac{1}{n}\big(\sum_{l=\lceil \frac{k}{u-k}\rceil} \frac{l}{n-1}(l(u-k)-u)+\sum_{l=1}^{\lceil \frac{k}{u-k}\rceil-1} \frac{l}{n-1}(k-u)\big)$	$\frac{1}{n}\big(k+\sum_{l=1}^{\lceil \frac{k}{u-k}\rceil-1}(k-l(u-k))\big)$
$\boldsymbol{\frac{n-1}{n} < \frac{k}{u} \le 1}$	$\frac{1}{n}\big(0+\sum_{l=1}^{n-1} \frac{l}{n-1}(k-u)\big)$	$\frac{1}{n}\big(k+\sum_{l=1}^{n-1}(k-l(u-k))\big)$

Table 1: Summary of the expressions of the Shapley value for the case $\lambda > 0$. Note that the expressions have not been fully simplified to highlight the effect of the various cases on the index set.

(i.e., $S = N$) are already satisfied for any value of the parameters. Moreover, we first consider the inequalities for coalitions S such that $i_p \notin S$ first, and afterwards the ones for which $i_p \in S$.

Case 2.1: $\boldsymbol{0 < \frac{k}{u} \le \frac{1}{2}}$. We first simplify the expressions for $\Phi_{-i_p}(c^+)$ and $\Phi_{i_p}(c^+)$ from Table 1:

$$\Phi_{-i_p}(c^+) = \frac{1}{n}\Big(\sum\nolimits_{l=1} \frac{l}{n-1}(l(u-k)-u)\Big) \tag{13a}$$

$$= -\frac{k}{n(n-1)} \tag{13b}$$

$$\Phi_{i_p}(c^+) = \frac{k}{n} \tag{13c}$$

We now test coalitional rationality, first for coalitions $S \subsetneq N$ such that $i_p \notin S$. In this case, the cost of coalition S equals $c_S^+ = 0$ and each inequality $\sum_{i \in S} \Phi_i(c^+) \le 0$ is equivalent to $\Phi_{-i_p}(c^+) \le 0$. As $k > 0$ and $n \ge 2$, $\Phi_{-i_p}(c^+) < 0$ and coalitional rationality is satisfied for these coalitions.

We now consider coalitions $S \subsetneq N$ such that $i_p \in S$. In this case, $c_S^+ = 0$ and the inequality to solve is the following:

$$\sum_{i \in S \setminus \{i_p\}} \Phi_i(c^+) + \Phi_{i_p}(c^+) \le 0 \tag{14a}$$

$$\Leftrightarrow \quad -\frac{(|S|-1)k}{n(n-1)} + \frac{k}{n} \le 0 \tag{14b}$$

$$\Leftrightarrow \quad -(|S|-1) + n - 1 \le 0 \tag{14c}$$

$$\Leftrightarrow \quad n - |S| \le 0 \tag{14d}$$

Since $S \subsetneq N$, (14d) cannot hold, which means that coalitional rationality is not satisfied for these coalitions S. Thus, we proved that $\Phi(c^+) \notin \mathfrak{C}(c^+)$ for $0 < \frac{k}{u} \le \frac{1}{2}$ as coalitional rationality failed for some coalitions. Note that the previous contradiction holds for all coalition sizes we need to test (i.e., $2 \leq |S| \leq n-1$).

Case 2.2: $\frac{1}{2} < \frac{k}{u} \leq \frac{n-1}{n}$. We follow the same procedure as in the previous case: we start by computing the expressions for $\Phi_{-i_p}(c^+)$ and $\Phi_{i_p}(c^+)$ from Table 1:

$$\begin{aligned}
\Phi_{-i_p}(c^+) &= \frac{1}{n}\big(\sum\nolimits_{l=\lceil \frac{k}{u-k}\rceil} \frac{l}{n-1}(l(u-k)-u) + \sum\nolimits_{l=1}^{\lceil \frac{k}{u-k}\rceil -1} \frac{l}{n-1}(k-u)\big) && (15a)\\
&= \frac{1}{n}\left(\frac{\left\lceil \frac{k}{u-k}\right\rceil}{n-1}(\left\lceil \frac{k}{u-k}\right\rceil (u-k)-u) + \frac{(k-u)}{n-1}\sum\nolimits_{l=1}^{\lceil \frac{k}{u-k}\rceil -1} l\right) && (15b)\\
&= \frac{1}{n(n-1)}\left(\left\lceil \frac{k}{u-k}\right\rceil \left(\left\lceil \frac{k}{u-k}\right\rceil (u-k)-u\right) + (k-u)\frac{\left\lceil \frac{k}{u-k}\right\rceil\left(\left\lceil \frac{k}{u-k}\right\rceil -1\right)}{2}\right) && (15c)\\
&= \frac{1}{2n(n-1)}\left\lceil \frac{k}{u-k}\right\rceil \left(\left\lceil \frac{k}{u-k}\right\rceil (u-k)-u-k\right) && (15d)\\
\Phi_{i_p}(c^+) &= \frac{1}{n}\big(k + \sum\nolimits_{l=1}^{\lceil \frac{k}{u-k}\rceil -1}(k-l(u-k))\big) && (15e)\\
&= \frac{1}{n}\left(k + (k-u)\frac{\left\lceil \frac{k}{u-k}\right\rceil\left(\left\lceil \frac{k}{u-k}\right\rceil -1\right)}{2} + k\left(\left\lceil \frac{k}{u-k}\right\rceil -1\right)\right) && (15f)\\
&= \frac{1}{n}\left\lceil \frac{k}{u-k}\right\rceil \left(\frac{(k-u)}{2}\left(\left\lceil \frac{k}{u-k}\right\rceil -1\right) + k\right) && (15g)\\
&= \frac{1}{2n}\left\lceil \frac{k}{u-k}\right\rceil \left((k-u)\left\lceil \frac{k}{u-k}\right\rceil + k + u\right) && (15h)
\end{aligned}$$

We now test coalitional rationality for coalitions S with $i_p \notin S$. We know that $c_S^+ = 0$ for these coalitions in this case. Thus, $\sum_{i\in S}\Phi_i(c^+) \leq c_S^+$ reduces to $\Phi_{-i_p}(c^+) \leq 0$, which can be rewritten as follows:

$$\begin{aligned}
& \Phi_{-i_p}(c^+) \leq 0 && (16a)\\
\Leftrightarrow \quad & \frac{1}{2n(n-1)}\left\lceil \frac{k}{u-k}\right\rceil \left(\left\lceil \frac{k}{u-k}\right\rceil (u-k)-u-k\right) \leq 0 && (16b)\\
\Leftrightarrow \quad & \left\lceil \frac{k}{u-k}\right\rceil \leq \frac{u+k}{u-k} && (16c)\\
\Leftrightarrow \quad & \left\lceil \frac{k}{u-k}\right\rceil - \frac{k}{u-k} \leq \frac{u}{u-k} && (16d)
\end{aligned}$$

As for all $0 < k < u$, $\left\lceil \frac{k}{u-k}\right\rceil - \frac{k}{u-k} < 1$ by definition of the ceiling operation and $\frac{u}{u-k} \geq 1$ as $u-k<u$, we have that (16d) is always satisfied for coalitions S such that $i_p \notin S$.

We now test coalitional rationality for coalitions S such that $i_p \in S$. From $\frac{1}{2} < \frac{k}{u} \le \frac{n-1}{n}$, we obtain $2 < \frac{u}{u-k} \le n$ and, therefore, we need to distinguish two further cases depending on the size $|S|$ of coalition S compared to $\frac{u}{u-k}$ as in the definition of c^+_S:

$$c^+_S = \begin{cases} 0 & i_p \in S \text{ and } |S| \ge \frac{u}{u-k} \ , \qquad (17\text{a}) \\ u - |S|(u-k) & i_p \in S \text{ and } |S| < \frac{u}{u-k} \ . \qquad (17\text{b}) \end{cases}$$

Therefore, we start with the case where $|S| \ge \frac{u}{u-k}$. It is important to note that, since $|S| \le n-1$, for values of $\frac{u}{u-k} > n-1$ this case will not occur as no coalition has such a size and we are left with the case that will be discussed later. The condition $\frac{u}{u-k} > n-1$ is equivalent to $\frac{k}{u} > \frac{n-2}{n-1}$. Note that $\frac{n-2}{n-1} < \frac{n-1}{n}$ for all $n > 2$.

$$(|S|-1)\Phi_{-i_p}(c^+) + \Phi_{i_p}(c^+) \le 0 \tag{18a}$$

$$\Leftrightarrow \quad \frac{|S|-1}{2n(n-1)} \left\lceil \frac{k}{u-k} \right\rceil \left(\left\lceil \frac{k}{u-k} \right\rceil (u-k) - u - k \right) + \frac{1}{2n} \left\lceil \frac{k}{u-k} \right\rceil \left((k-u) \left\lceil \frac{k}{u-k} \right\rceil + k + u \right) \le 0 \tag{18b}$$

$$\Leftrightarrow \quad \frac{n-|S|}{2(n-1)} \left\lceil \frac{k}{u-k} \right\rceil \left(k \left\lceil \frac{k}{u-k} \right\rceil + k - u \left\lceil \frac{k}{u-k} \right\rceil + u \right) \le 0 \tag{18c}$$

$$\Leftrightarrow \quad \left\lceil \frac{k}{u-k} \right\rceil (k-u) + u + k \le 0 \tag{18d}$$

$$\Leftrightarrow \quad u + k \le (u-k) \left\lceil \frac{k}{u-k} \right\rceil \tag{18e}$$

$$\Leftrightarrow \quad \frac{u}{u-k} \le \left\lceil \frac{k}{u-k} \right\rceil - \frac{k}{u-k} \tag{18f}$$

As $1 \le \frac{u}{u-k}$ and $\left\lceil \frac{k}{u-k} \right\rceil - \frac{k}{u-k} < 1$ by definition of the ceiling operation, (18f) does not hold, which means that coalitional rationality is not satisfied in this case. Thus, for $\frac{1}{2} < \frac{k}{u} \le \frac{n-2}{n-1}$, we have $\Phi(c^+) \notin \mathfrak{C}(c^+)$.

We can now test coalitional rationality for the case $|S| < \frac{u}{u-k}$ when $\frac{n-2}{n-1} < \frac{k}{u} \le \frac{n-1}{n}$, as for the case $\frac{1}{2} < \frac{k}{u} \le \frac{n-2}{n-1}$ coalitional rationality was not achieved. We note that $\frac{u}{u-k} > n-1$ is equivalent to $\frac{k}{u-k} > n-2$, and that $\frac{k}{u} \le \frac{n-1}{n}$ is equivalent to $\frac{u}{u-k} \le n$, from which it follows that $n-2 < \frac{k}{u-k} \le n-1$ and $\left\lceil \frac{k}{u-k} \right\rceil = n-1$. We can then solve the following

$$(|S|-1)\Phi_{-i_p}(c^+) + \Phi_{i_p}(c^+) \le u - |S|(u-k) \tag{19a}$$

$$\Leftrightarrow \quad \frac{|S|-1}{2n(n-1)} \left\lceil \frac{k}{u-k} \right\rceil \left(\left\lceil \frac{k}{u-k} \right\rceil (u-k) - u - k \right) + \frac{1}{2n} \left\lceil \frac{k}{u-k} \right\rceil \left((k-u) \left\lceil \frac{k}{u-k} \right\rceil + k + u \right) \le u - |S|(u-k) \tag{19b}$$

$$\Leftrightarrow \quad \frac{1}{2n}(kn^2 - kn|S| - 2k|S| - n^2u + n|S|u + 2nu - 2u) \le 0 \tag{19c}$$

$$\Leftrightarrow \quad kn^2 + |S|((3n-2)u - 3kn) - n^2u \le 0 \tag{19d}$$

At this point, we note that this expression is linear in $|S|$ which takes values in $\{2,3,\dots,n-1\}$. Thus, depending on the sign of the coefficient $(3n-2)u-3kn$, it will be sufficient to test this expression for $|S|=2$ or $|S|=n-1$ as the maximum of the expression on the left must be achieved at one of these two extremes. We test for positivity of the coefficient:

$$(3n-2)u-3kn>0 \tag{20a}$$

$$\Leftrightarrow \quad \frac{k}{u}>\frac{3n-2}{3n} \tag{20b}$$

Note that $\frac{3n-2}{3n}>\frac{n-1}{n}$ and, therefore, the coefficient $(3n-2)u-3kn$ is positive for all values of k and u such that $\frac{n-2}{n-1}<\frac{k}{u}\leq\frac{n-1}{n}$. We can then replace $|S|=n-1$ in (19d) and obtain:

$$kn(3-2n)+u(2-5n+2n^2)\leq 0 \tag{21a}$$

$$\Leftrightarrow \quad (2-5n+2n^2)\leq\frac{k}{u}(2n^2-3n) \tag{21b}$$

$$\Leftrightarrow \quad \frac{2n^2-5n+2}{2n^2-3n}\leq\frac{k}{u} \tag{21c}$$

$$\Leftrightarrow \quad \frac{(2n-1)(n-2)}{2n^2-3n}\leq\frac{k}{u} \tag{21d}$$

Because we have $\frac{(2n-1)(n-2)}{2n^2-3n}>\frac{n-2}{n-1}$ and $\frac{(2n-1)(n-2)}{2n^2-3n}<\frac{n-1}{n}$ for all $n>2$, we can conclude that, for values of k and u such that $\frac{(2n-1)(n-2)}{2n^2-3n}\leq\frac{k}{u}\leq\frac{n-1}{n}$, the Shapley value is stable, i.e., $\Phi(c^+)\in\mathfrak{C}(c^+)$.

Case 2.3: $\frac{n-1}{n}<\frac{k}{u}\leq 1$. Again, we follow the same procedure as above: we start by computing the expressions for $\Phi_{-i_p}(c^+)$ and $\Phi_{i_p}(c^+)$ from Table 1:

$$\Phi_{-i_p}(c^+)=\frac{1}{n}\big(0+\sum\nolimits_{l=1}^{n-1}\frac{l}{n-1}(k-u)\big) \tag{22a}$$

$$=\frac{(k-u)}{n(n-1)}\frac{n(n-1)}{2} \tag{22b}$$

$$=\frac{k-u}{2} \tag{22c}$$

$$\Phi_{i_p}(c^+)=\frac{1}{n}\big(k+\sum\nolimits_{l=1}^{n-1}(k-l(u-k))\big) \tag{22d}$$

$$=\frac{1}{n}\big(k+(n-1)k-(u-k)\frac{n(n-1)}{2}\big) \tag{22e}$$

$$=k+(k-u)\frac{n-1}{2} \tag{22f}$$

We now check coalitional rationality starting with coalitions S such that $i_p\notin S$. In this case, $c_S^+=0$, and the inequality to test is:

$$|S|\Phi_{-i_p}(c^+) \leq 0 \tag{23a}$$

$$\Leftrightarrow \quad \Phi_{-i_p}(c^+) \leq 0 \tag{23b}$$

$$\Leftrightarrow \quad \frac{k-u}{2} \leq 0 \tag{23c}$$

Since $k < u$, (23c) holds true, which means that coalitional rationality is satisfied for these coalitions.

We now test coalitional rationality for coalitions S such that $i_p \in S$. In this case, we have $c_S^+ = u - |S|(u-k)$ and

$$(|S|-1)\Phi_{-i_p}(c^+) + \Phi_{i_p}(c^+) \leq c_S^+ \tag{24a}$$

$$\Leftrightarrow \quad (|S|-1)\frac{k-u}{2} + k + (k-u)\frac{n-1}{2} \leq u - |S|(u-k) \tag{24b}$$

$$\Leftrightarrow \quad (k-u)(n-2) \leq 0 \tag{24c}$$

Since $k < u$, (24c) holds true, which means that we have shown that $\Phi(c^+) \in \mathfrak{C}(c^+)$ for $\frac{n-1}{n} < \frac{k}{u} \leq 1$.

Case	Result
$\boldsymbol{0 < \frac{k}{u} \leq \frac{1}{2}}$	$\Phi(c^+) \notin \mathfrak{C}(c^+)$
$\boldsymbol{\frac{1}{2} < \frac{k}{u} \leq \frac{n-1}{n}}$	$\Phi(c^+) \in \mathfrak{C}(c^+)$ if and only if $\frac{(2n-1)(n-2)}{2n^2-3n} \leq \frac{k}{u} \leq \frac{n-1}{n}$
$\boldsymbol{\frac{n-1}{n} < \frac{k}{u} \leq 1}$	$\Phi(c^+) \in \mathfrak{C}(c^+)$

Table 2: Summary of the findings in the three cases for $\lambda > 0$.

A summary of the findings in the previous three cases is presented in Table 2, which can be combined into

$$\Phi(c^+) \in \mathfrak{C}(c^+) \text{ if and only if } \frac{(2n-1)(n-2)}{2n^2-3n} \leq \frac{k}{u} \leq 1. \tag{25}$$

We can now move to consider the third and last case.

Case 3: $\boldsymbol{\lambda < 0}$. Our discussion of this case follows the same structure as the one of Case 2.

For a given coalition $S \subseteq N$ and $\lambda \in [-c, 0]$, the cost $c(\lambda)$ of coalition S is given as follows:

$$c_S(\lambda) = \begin{cases} |S|kc & i_p \notin S \quad (26a) \\ |S|k(c+\lambda) & i_p \in S \text{ and } |S| \leq \frac{u}{k} \quad (26b) \\ u(c+\lambda) + c(|S|k-u) & i_p \in S \text{ and } |S| > \frac{u}{k} \quad (26c) \end{cases}$$

The conditions $|S| \le \frac{u}{k}$ and $|S| > \frac{u}{k}$ in (26b) and (26c) can be rewritten as $u \ge |S|k$ and $u < |S|k$, respectively. They represent the two cases where all the $|S|k$ orders can either be transported on arc r_{i_p} or not. Indeed, as $\lambda < 0$, arc r_{i_p} is used first in any optimal solution of the minimum cost flow problem for coalition S.

We rewrite $c(\lambda)$ as $c(\lambda) = c^0 + \lambda c^-$, where

$$c^-_S = \begin{cases} 0 & i_p \notin S\ , \qquad (27a) \\ |S|k & i_p \in S \text{ and } |S| \le \frac{u}{k}\ , \qquad (27b) \\ u & i_p \in S \text{ and } |S| > \frac{u}{k}\ . \qquad (27c) \end{cases}$$

We now observe that, as in Case 2, this decomposition simplifies checking the stability of the Shapley value. Indeed, using linearity of the Shapley value, we can write $\Phi_i(\lambda) = \Phi_i(c^0) + \lambda\Phi_i(c^-)$ for each player $i \in N$ and, for each coalition $S \subseteq N$, we can rewrite the inequality $\sum_{i\in S} \Phi_i(\lambda) \le c_S(\lambda)$ as follows:

$$\sum_{i\in S} \Phi_i(\lambda) \le c(\lambda) \qquad (28a)$$

$$\Leftrightarrow \quad \sum_{i\in S} \left(\Phi_i(c^0) + \lambda\Phi_i(c^-)\right) \le c^0_S + \lambda c^-_S \qquad (28b)$$

$$\Leftrightarrow \quad \sum_{i\in S} \Phi_i(c^0) + \lambda \sum_{i\in S} \Phi_i(c^-) \le c^0_S + \lambda c^-_S \qquad (28c)$$

$$\Leftrightarrow \quad \lambda \sum_{i\in S} \Phi_i(c^-) \le \lambda c^-_S \qquad \left(\text{as } \sum_{i\in S} \Phi_i(c^0) = c^0_S\right) \qquad (28d)$$

$$\Leftrightarrow \quad \sum_{i\in S} \Phi_i(c^-) \ge c^-_S \qquad (\text{as } \lambda < 0) \qquad (28e)$$

Therefore, the Shapley value is stable in the game $c(\lambda)$ for $\lambda < 0$ if and only if $\sum_{i\in S} \Phi_i(c^-) \ge c^-_S$ for all coalitions $S \subseteq N$. As in Case 2, note that, in (28), we have obtained both independence of λ and a much simpler game to analyze.

We now compute the marginal cost for each player $i \ne i_p$:

$$c^-_{S\cup\{i\}} - c^-_S = \begin{cases} 0 & i_p \notin S \qquad (29a) \\ k & i_p \in S \text{ and } |S| + 1 \le \frac{u}{k} \qquad (29b) \\ u - |S|k & i_p \in S \text{ and } \frac{u}{k} - 1 < |S| \le \frac{u}{k} \qquad (29c) \\ 0 & i_p \in S \text{ and } |S| > \frac{u}{k} \qquad (29d) \end{cases}$$

For $i = i_p$, we have $i_p \notin S$, and the marginal cost equals

$$c^-_{S\cup\{i_p\}} - c^-_S = \begin{cases} (|S|+1)k & |S| + 1 \le \frac{u}{k} \qquad (30a) \\ u & \text{and } |S| + 1 > \frac{u}{k}\ . \qquad (30b) \end{cases}$$

We now compute the expressions for $\Phi_{-i_p}(c^-)$ and $\Phi_{i_p}(c^-)$.

For $i \neq i_p$:

$$\Phi_i(c^-) = \frac{1}{n} \sum_{S \subseteq N \setminus \{i\}} \binom{n-1}{|S|}^{-1} (c^-_{S \cup \{i\}} - c^-_S) \tag{31a}$$

$$= \frac{1}{n} \left(\sum_{\substack{S \subseteq N \setminus \{i\}: \\ i_p \in S, \\ |S| \le \frac{u}{k} - 1}} \binom{n-1}{|S|}^{-1} (c^-_{S \cup \{i\}} - c^-_S) + \sum_{\substack{S \subseteq N \setminus \{i\}: \\ i_p \in S, \\ \frac{u}{k} - 1 < |S| \le \frac{u}{k}}} \binom{n-1}{|S|}^{-1} (c^-_{S \cup \{i\}} - c^-_S) \right) \tag{31b}$$

$$= \frac{1}{n} \left(\sum_{\substack{l=1,\dots,n-1: \\ l \le \frac{u}{k} - 1}} \sum_{\substack{S \subseteq N \setminus \{i\}: \\ i_p \in S, \\ |S| = l}} \binom{n-1}{l}^{-1} k + \sum_{\substack{l=1,\dots,n-1: \\ \frac{u}{k} - 1 < l \le \frac{u}{k}}} \sum_{\substack{S \subseteq N \setminus \{i\}: \\ i_p \in S, \\ |S| = l}} \binom{n-1}{l}^{-1} (u - lk) \right) \tag{31c}$$

$$= \frac{1}{n} \left(\sum_{\substack{l=1,\dots,n-1: \\ l \le \frac{u}{k} - 1}} \binom{n-1}{l}^{-1} \binom{n-2}{l-1} k + \sum_{\substack{l=1,\dots,n-1: \\ \frac{u}{k} - 1 < l \le \frac{u}{k}}} \binom{n-1}{l}^{-1} \binom{n-2}{l-1} (u - lk) \right) \tag{31d}$$

$$= \frac{1}{n} \left(\sum_{\substack{l=1,\dots,n-1: \\ l \le \frac{u}{k} - 1}} k \frac{l}{n-1} + \sum_{\substack{l=1,\dots,n-1: \\ \frac{u}{k} - 1 < l \le \frac{u}{k}}} (u - lk) \frac{l}{n-1} \right) \tag{31e}$$

For $i = i_p$:

$$\Phi_{i_p}(c^-) = \frac{1}{n} \sum_{S \subseteq N \setminus \{i_p\}} \binom{n-1}{|S|}^{-1} (c^-_{S \cup \{i_p\}} - c^-_S) \tag{32a}$$

$$= \frac{1}{n} \left(\sum_{\substack{S \subseteq N \setminus \{i_p\}: \\ |S| \le \frac{u}{k} - 1}} \binom{n-1}{|S|}^{-1} (c^-_{S \cup \{i_p\}} - c^-_S) + \sum_{\substack{S \subseteq N \setminus \{i_p\}: \\ |S| > \frac{u}{k} - 1}} \binom{n-1}{|S|}^{-1} (c^-_{S \cup \{i_p\}} - c^-_S) \right) \tag{32b}$$

$$= \frac{1}{n} \left(\sum_{\substack{l=0,1,\dots,n-1: \\ l \le \frac{u}{k} - 1}} \sum_{\substack{S \subseteq N \setminus \{i_p\}: \\ |S| = l}} \binom{n-1}{l}^{-1} (l+1) k + \sum_{\substack{l=0,1,\dots,n-1: \\ l > \frac{u}{k} - 1}} \sum_{\substack{S \subseteq N \setminus \{i_p\}: \\ |S| = l}} \binom{n-1}{l}^{-1} u \right) \tag{32c}$$

$$= \frac{1}{n} \left(\sum_{\substack{l=0,1,\dots,n-1: \\ l \le \frac{u}{k} - 1}} \binom{n-1}{l}^{-1} \binom{n-1}{l} (l+1) k + \sum_{\substack{l=0,1,\dots,n-1: \\ l > \frac{u}{k} - 1}} \binom{n-1}{l}^{-1} \binom{n-1}{l} u \right) \tag{32d}$$

$$= \frac{1}{n}\left(\sum_{\substack{l=0,1,\ldots,n-1:\\ l\le \frac{u}{k}-1}} (l+1)k + \sum_{\substack{l=0,1,\ldots,n-1:\\ l> \frac{u}{k}-1}} u\right) \tag{32e}$$

It is possible to rewrite the expressions in (32e) and in (31e) further by analysing the index sets of the two summations as, depending on the value of $\frac{k}{u}$, they can be empty or not. We distinguish four cases that are presented in Table 3 and discuss them now.

In case $0 < \frac{k}{u} \le \frac{1}{n}$, we have $\frac{u}{k} - 1 \ge n-1$, and the index set $\{l = 1,\ldots,n-1 : l \le \frac{u}{k}-1\}$ equals $\{l = 1,\ldots,n-1\}$, while the index set $\{l = 1,\ldots,n-1 : \frac{u}{k}-1 < l \le \frac{u}{k}\}$ is empty and the summation with this index set is zero. Moreover, the index set $\{l = 0,1,\ldots,n-1 : l \le \frac{u}{k}-1\}$ equals $\{l = 0,1,\ldots,n-1\}$, while the index set $\{l = 1,\ldots,n-1 : l > \frac{u}{k}-1\}$ is empty.

In case $\frac{1}{n} < \frac{k}{u} \le \frac{1}{n-1}$, we have $n-2 \le \frac{u}{k}-1 < n-1$, and the index set $\{l = 1,\ldots,n-1 : l \le \frac{u}{k}-1\}$ equals $\{l = 1,\ldots,\lfloor \frac{u}{k}-1\rfloor\}$, while the index set $\{l = 1,\ldots,n-1 : \frac{u}{k}-1 < l \le \frac{u}{k}\}$ equals $\{l = n-1\}$. Moreover, the index set $\{l = 0,1,\ldots,n-1 : l \le \frac{u}{k}-1\}$ equals $\{l = 0,1,\ldots,\lfloor \frac{u}{k}-1\rfloor\}$, while the index set $\{l = 1,\ldots,n-1 : l > \frac{u}{k}-1\}$ reduces to the singleton $\{l = \lceil \frac{u}{k}-1\rceil\}$.

In case $\frac{1}{n-1} < \frac{k}{u} \le \frac{1}{2}$, we have $1 \le \frac{u}{k}-1 < n-2$, and the index set $\{l = 1,\ldots,n-1 : l \le \frac{u}{k}-1\}$ equals $\{l = 1,\ldots,\lfloor \frac{u}{k}-1\rfloor\}$ as in the previous case, while the index set $\{l = 1,\ldots,n-1 : \frac{u}{k}-1 < l \le \frac{u}{k}\}$ equals $\{l = \lfloor \frac{u}{k}\rfloor\}$. Moreover, the index set $\{l = 0,1,\ldots,n-1 : l \le \frac{u}{k}-1\}$ equals $\{l = 0,1,\ldots,\lfloor \frac{u}{k}-1\rfloor\}$ as in the previous case, while the index set $\{l = 1,\ldots,n-1 : l > \frac{u}{k}-1\}$ equals $\{l = \lfloor \frac{u}{k}\rfloor, \lfloor \frac{u}{k}\rfloor+1,\ldots,n-1\}$.

In case $\frac{1}{2} < \frac{k}{u} \le 1$, we have $0 \le \frac{u}{k}-1 < 1$, and the index set $\{l = 1,\ldots,n-1 : l \le \frac{u}{k}-1\}$ is empty, while the index set $\{l = 1,\ldots,n-1 : \frac{u}{k}-1 < l \le \frac{u}{k}\}$ equals $\{l = \lfloor \frac{u}{k}\rfloor\}$. Moreover, the index set $\{l = 0,1,\ldots,n-1 : l \le \frac{u}{k}-1\}$ reduces to the singleton $\{l = \lfloor \frac{u}{k}\rfloor\}$, while the index set $\{l = 1,\ldots,n-1 : l > \frac{u}{k}-1\}$ equals $\{l = 1,\ldots,n-1\}$.

Case 3.1: $\boldsymbol{0 < \frac{k}{u} \le \frac{1}{n}}$. We start by computing the expressions for $\Phi_{-i_p}(c^-)$ and $\Phi_{i_p}(c^-)$ from Table 3:

$$\Phi_{-i_p}(c^-) = \frac{1}{n}\sum\nolimits_{l=1}^{n-1} k\frac{l}{n-1} \tag{33a}$$

$$= \frac{k}{n(n-1)}\frac{n(n-1)}{2} \tag{33b}$$

$$= \frac{k}{2} \tag{33c}$$

$$\Phi_{i_p}(c^-) = \frac{1}{n}\sum\nolimits_{l=0}^{n-1}(l+1)k \tag{33d}$$

$$= \frac{k}{n}\sum\nolimits_{l=1}^{n} l \tag{33e}$$

$$= \frac{k}{n}\frac{n(n+1)}{2} \tag{33f}$$

Case	$\Phi_{-i_p}(c^-)$	$\Phi_{i_p}(c^-)$
$\mathbf{0<\frac{k}{u}\leq\frac{1}{n}}$	$\frac{1}{n}\sum_{l=1}^{n-1}k\frac{l}{n-1}$	$\frac{1}{n}\sum_{l=0}^{n-1}(l+1)k$
$\mathbf{\frac{1}{n}<\frac{k}{u}\leq\frac{1}{n-1}}$	$\frac{1}{n}\Big(\sum_{l=1}^{\lfloor\frac{u}{k}-1\rfloor}k\frac{l}{n-1}+\sum_{l=n-1}(u-lk)\frac{l}{n-1}\Big)$	$\frac{1}{n}\Big(\sum_{l=0}^{\lfloor\frac{u}{k}-1\rfloor}(l+1)k+\sum_{l=\lceil\frac{u}{k}-1\rceil}u\Big)$
$\mathbf{\frac{1}{n-1}<\frac{k}{u}\leq\frac{1}{2}}$	$\frac{1}{n}\Big(\sum_{l=1}^{\lfloor\frac{u}{k}-1\rfloor}k\frac{l}{n-1}+\sum_{l=\lfloor\frac{u}{k}\rfloor}(u-lk)\frac{l}{n-1}\Big)$	$\frac{1}{n}\Big(\sum_{l=0}^{\lfloor\frac{u}{k}-1\rfloor}(l+1)k+\sum_{l=\lfloor\frac{u}{k}\rfloor}^{n-1}u\Big)$
$\mathbf{\frac{1}{2}<\frac{k}{u}\leq 1}$	$\frac{1}{n}\Big(0+\sum_{l=\lfloor\frac{u}{k}\rfloor}(u-lk)\frac{l}{n-1}\Big)$	$\frac{1}{n}\Big(\sum_{l=0}(l+1)k+\sum_{l=1}^{n-1}u\Big)$

Table 3: Summary of the expressions of the Shapley value for the case $\lambda<0$. Note that the expressions have not been fully simplified to highlight the effect of the various cases on the index set.

$$=\frac{k(n+1)}{2} \tag{33g}$$

We now test coalitional rationality of the Shapley value by using the equivalent expression (28e) for all coalitions $S\subseteq N$ starting with coalitions S such that $i_p\notin S$. In this case, $c_S^-=0$, and the inequality to test is the following:

$$|S|\Phi_{-i_p}(c^-)\geq 0 \tag{34a}$$

$$\Leftrightarrow\quad \Phi_{-i_p}(c^-)\geq 0 \tag{34b}$$

$$\Leftrightarrow\quad \frac{k}{2}\geq 0 \tag{34c}$$

Since $k>0$, (34c) holds true for all coalitions S such that $i_p\notin S$. We now test coalitional rationality for coalitions S such that $i_p\in S$. In this case, $c_S^-=|S|k$, and the inequality to test is

$$(|S|-1)\Phi_{-i_p}(c^-)+\Phi_{i_p}(c^-)\geq|S|k \tag{35a}$$

$$\Leftrightarrow\quad (|S|-1)\frac{k}{2}+\frac{k}{2}(n+1)\geq|S|k \tag{35b}$$

$$\Leftrightarrow\quad kn\geq|S|k \tag{35c}$$

$$\Leftrightarrow\quad n\geq|S| \tag{35d}$$

Since $S \subseteq N$, (35d) holds true and, overall, we obtain that the Shapley value $\Phi(c(\lambda))$ is stable in this case.

Case 3.2: $\boldsymbol{\frac{1}{n} < \frac{k}{u} \le \frac{1}{n-1}}$. We start by computing the expressions for $\Phi_{-i_p}(c^-)$ and $\Phi_{i_p}(c^-)$ from Table 3. Note that, in this case, $\lfloor \frac{u}{k} - 1 \rfloor = n-2$. Indeed, the condition defining this case is equivalent to $n-1 \le \frac{u}{k} < n$ and, if $\frac{u}{k} \in \mathbb{N}$, then $\frac{u}{k} = n-1$. If $\frac{u}{k} \notin \mathbb{N}$ instead, then $\lceil \frac{u}{k} - 1 \rceil = \lceil \frac{u}{k} \rceil - 1 = n-2$. This makes the limit of the first summations in the expressions of $\Phi_{-i_p}(c^-)$ and $\Phi_{i_p}(c^-)$ explicit.

$$\Phi_{-i_p}(c^-) = \frac{1}{n}\left(\sum_{l=1}^{\lfloor \frac{u}{k}-1 \rfloor} k\frac{l}{n-1} + \sum_{l=n-1}(u-lk)\frac{l}{n-1}\right) \tag{36a}$$

$$= \frac{k}{n(n-1)}\sum_{l=1}^{n-2} l + \frac{1}{n}(u-(n-1)k)\frac{n-1}{n-1} \tag{36b}$$

$$= \frac{k}{n(n-1)}\frac{(n-1)(n-2)}{2} + \frac{u-(n-1)k}{n} \tag{36c}$$

$$= \frac{-nk+2u}{2n} \tag{36d}$$

$$\Phi_{i_p}(c^-) = \frac{1}{n}\left(\sum_{l=0}^{\lfloor \frac{u}{k}-1 \rfloor}(l+1)k + \sum_{l=\lceil \frac{u}{k}-1 \rceil} u\right) \tag{36e}$$

$$= \frac{k}{n}\sum_{l=1}^{n-1} l + \frac{u}{n} \tag{36f}$$

$$= \frac{k}{n}\frac{n(n-1)}{2} + \frac{u}{n} \tag{36g}$$

$$= \frac{n(n-1)k+2u}{2n} \tag{36h}$$

Similarly to the previous case, we test coalitional rationality starting with coalitions S such that $i_p \notin S$. In this case, $c_S^- = 0$, and the inequality to test is

$$|S|\Phi_{-i_p}(c^-) \ge 0 \tag{37a}$$

$$\Leftrightarrow \quad \Phi_{-i_p}(c^-) \ge 0 \tag{37b}$$

$$\Leftrightarrow \quad \frac{2u-nk}{2n} \ge 0 \tag{37c}$$

$$\Leftrightarrow \quad 2u \ge nk \tag{37d}$$

$$\Leftrightarrow \quad \frac{k}{u} \le \frac{2}{n} \tag{37e}$$

Since the case-defining condition $\frac{k}{u} \le \frac{1}{n-1}$ implies that $\frac{k}{u} \le \frac{2}{n}$ (as $\frac{1}{n-1} \le \frac{2}{n}$ for $n \ge 2$), (37e) holds true.

We now consider coalitions S such that $i_p \in S$. In this case, $c_S^- = |S|k$, and the inequality to test is

$$(|S|-1)\Phi_{-i_p}(c^-)+\Phi_{i_p}(c^-)\geq |S|k \tag{38a}$$

$$\Leftrightarrow \quad (|S|-1)\frac{2u-nk}{2n}+\frac{n(n-1)k+2u}{2n}\geq |S|k \tag{38b}$$

$$\Leftrightarrow \quad n^2k-3n|S|k+2|S|u\geq 0 \tag{38c}$$

$$\Leftrightarrow \quad |S|(2u-3nk)\geq -n^2k \tag{38d}$$

as $(2u - 3nk) < 0$ and the inequality should hold for all $|S| = 1,2,\ldots,n-1$, it suffices to consider $|S|=n-1$:

$$\Leftrightarrow \quad (n-1)(2u-3nk)\geq -n^2k \tag{38e}$$

$$\Leftrightarrow \quad 2\frac{u}{k}(n-1)\geq 2n^2-3n \tag{38f}$$

$$\Leftrightarrow \quad \frac{k}{u}\leq\frac{2n-2}{2n^2-3n} \tag{38g}$$

Since $\frac{1}{n}<\frac{2n-2}{2n^2-3n}\leq\frac{1}{n-1}$ for all $n\geq 2$, (38g) defines a new condition for stability in this case.

Combining the results just obtained yields $\Phi(c(\lambda))\in\mathfrak{C}(\lambda)$ for $\frac{1}{n}<\frac{k}{u}\leq\frac{2n-2}{2n^2-3n}$.

Case 3.3: $\frac{1}{n-1}<\frac{k}{u}\leq\frac{1}{2}$. We start by computing the expressions for $\Phi_{-i_p}(c^-)$ and $\Phi_{i_p}(c^-)$ from Table 3:

$$\Phi_{-i_p}(c^-)=\frac{1}{n}\left(\sum\nolimits_{l=1}^{\left\lfloor\frac{u}{k}-1\right\rfloor}k\frac{l}{n-1}+\sum\nolimits_{l=\left\lfloor\frac{u}{k}\right\rfloor}(u-lk)\frac{l}{n-1}\right) \tag{39a}$$

$$=\frac{k}{n(n-1)}\frac{\left\lfloor\frac{u}{k}\right\rfloor\left\lfloor\frac{u}{k}-1\right\rfloor}{2}+\frac{\left(u-\left\lfloor\frac{u}{k}\right\rfloor k\right)\left\lfloor\frac{u}{k}\right\rfloor}{n(n-1)} \tag{39b}$$

$$=\frac{k}{2n(n-1)}\left[\left\lfloor\frac{u}{k}\right\rfloor\left(\left\lfloor\frac{u}{k}\right\rfloor-1\right)+2\left\lfloor\frac{u}{k}\right\rfloor\left(\frac{u}{k}-\left\lfloor\frac{u}{k}\right\rfloor\right)\right] \tag{39c}$$

$$\Phi_{i_p}(c^-)=\frac{1}{n}\left(\sum\nolimits_{l=0}^{\left\lfloor\frac{u}{k}-1\right\rfloor}(l+1)k+\sum\nolimits_{l=\left\lfloor\frac{u}{k}\right\rfloor}^{n-1}u\right) \tag{39d}$$

$$=\frac{k}{n}\sum\nolimits_{l=2}^{\left\lfloor\frac{u}{k}\right\rfloor}l+\frac{u}{n}\left(n-\left\lfloor\frac{u}{k}\right\rfloor\right) \tag{39e}$$

$$=\frac{k}{2n}\left\lfloor\frac{u}{k}\right\rfloor\left(\left\lfloor\frac{u}{k}\right\rfloor-1\right)+u-\frac{u}{n}\left\lfloor\frac{u}{k}\right\rfloor \tag{39f}$$

We now check whether coalitional rationality holds for all $S\subseteq N$ starting with coalitions S such that $i_p\notin S$. In this case,$c_S^-=0$, and the inequality to test is

$$|S|\Phi_{-i_p}(c^-)\geq 0 \tag{40a}$$

$$\Leftrightarrow \quad \Phi_{-i_p}(c^-)\geq 0 \tag{40b}$$

$$\Leftrightarrow \quad \frac{k}{2n(n-1)}\left[\left\lfloor\frac{u}{k}\right\rfloor\left(\left\lfloor\frac{u}{k}\right\rfloor-1\right)+2\left\lfloor\frac{u}{k}\right\rfloor\left(\frac{u}{k}-\left\lfloor\frac{u}{k}\right\rfloor\right)\right]\geq 0 \tag{40c}$$

$$\Leftrightarrow \quad \left(\left\lfloor\frac{u}{k}\right\rfloor-1\right)+2\left(\frac{u}{k}-\left\lfloor\frac{u}{k}\right\rfloor\right)\geq 0 \tag{40d}$$

Since $\frac{u}{k}\geq 2$ in this case and $\frac{u}{k}-\left\lfloor\frac{u}{k}\right\rfloor\geq 1$ by definition of the floor operation, (40d) holds true.

We now consider coalitional rationality for coalitions S such that $i_p\in S$. In this case, the value of c^-S depends on whether $|S|\leq\frac{u}{k}$ or not. We treat only the case $|S|>\frac{u}{k}$ as it will suffice to prove that the Shapley value is not stable in this case. For $|S|>\frac{u}{k}$, we have $c^-_S=u$ and the inequality to test is

$$(|S|-1)\Phi_{-i_p}(c^-)+\Phi_{i_p}(c^-)\geq u \tag{41a}$$

$$\Leftrightarrow \quad (|S|-1)\frac{k}{2n(n-1)}\left[\left\lfloor\frac{u}{k}\right\rfloor\left(\left\lfloor\frac{u}{k}\right\rfloor-1\right)+2\left\lfloor\frac{u}{k}\right\rfloor\left(\frac{u}{k}-\left\lfloor\frac{u}{k}\right\rfloor\right)\right]+ \tag{41b}$$

$$\frac{k}{2n}\left\lfloor\frac{u}{k}\right\rfloor\left(\left\lfloor\frac{u}{k}\right\rfloor-1\right)+u-\frac{u}{n}\left\lfloor\frac{u}{k}\right\rfloor\geq u \tag{41c}$$

$$\Leftrightarrow \quad \frac{|S|-1}{n-1}\left[\left\lfloor\frac{u}{k}\right\rfloor\left(\left\lfloor\frac{u}{k}\right\rfloor-1\right)+2\left\lfloor\frac{u}{k}\right\rfloor\left(\frac{u}{k}-\left\lfloor\frac{u}{k}\right\rfloor\right)\right]+\left\lfloor\frac{u}{k}\right\rfloor\left(\left\lfloor\frac{u}{k}\right\rfloor-2\frac{u}{k}\right)+\left\lfloor\frac{u}{k}\right\rfloor\geq 0 \tag{41d}$$

we test the case $|S|=n-1$ and obtain that

$$\Leftrightarrow \quad \left\lfloor\frac{u}{k}\right\rfloor-2\frac{u}{k}+1\geq 0 \tag{41e}$$

$$\Leftrightarrow \quad \left\lfloor\frac{u}{k}\right\rfloor-\frac{u}{k}+1\geq\frac{u}{k} \tag{41f}$$

Since $\left\lfloor\frac{u}{k}\right\rfloor-\frac{u}{k}+1\leq 1$ and $\frac{u}{k}\geq 2$, (41f) does not hold, and we obtain that the Shapley value is not stable in this case.

Case 3.4: $\boldsymbol{\frac{1}{2}<\frac{k}{u}\leq 1}$. We start by computing the expressions for $\Phi_{-i_p}(c^-)$ and $\Phi_{i_p}(c^-)$ from Table 3:

$$\Phi_{-i_p}(c^-)=\frac{1}{n}\sum\nolimits_{l=\left\lfloor\frac{u}{k}\right\rfloor}(u-lk)\frac{l}{n-1} \tag{42a}$$

$$=\frac{u-k}{n(n-1)} \tag{42b}$$

$$\Phi_{i_p}(c^-)=\frac{1}{n}\left(\sum\nolimits_{l=0}(l+1)k+\sum\nolimits_{l=1}^{n-1}u\right) \tag{42c}$$

$$=\frac{k}{n}+\frac{u}{n}(n-1) \tag{42d}$$

$$=\frac{k+(n-1)u}{n} \tag{42e}$$

We now test coalitional rationality of the Shapley value for all coalitions $S\subseteq N$ starting with coalitions S such that $i_p\notin S$. In this case, $c^-_S=0$, and the inequality to test is

$$
\begin{aligned}
& |S|\Phi_{-i_p}(c^-) \geq 0 && \text{(43a)}\\
\Leftrightarrow \quad & \Phi_{-i_p}(c^-) \geq 0 && \text{(43b)}\\
\Leftrightarrow \quad & \frac{u-k}{n(n-1)} \geq 0 && \text{(43c)}
\end{aligned}
$$

Since $k < u$, (43c) holds true.

We now consider coalitions S such that $i_p \in S$. Moreover, we consider the case where $|S| > 1$ as, for $|S| = 1$, we would be testing individual rationality of the Shapley value, which is always satisfied. Under these assumptions, we have $c^-_S = u$ and the inequality to test is

$$
\begin{aligned}
& (|S|-1)\Phi_{-i_p}(c^-) + \Phi_{i_p}(c^-) \geq u && \text{(44a)}\\
\Leftrightarrow \quad & \frac{(|S|-1)(u-k)}{n(n-1)} + \frac{k+(n-1)u}{n} \geq u && \text{(44b)}\\
\Leftrightarrow \quad & (u-k)(|S|-n) \geq 0 && \text{(44c)}
\end{aligned}
$$

Since $S \subsetneq N$, (44c) does not hold, meaning that the Shapley value is not stable in this case.

Case	Result
$\mathbf{0} < \frac{\boldsymbol{k}}{\boldsymbol{u}} \leq \frac{\mathbf{1}}{\boldsymbol{n}}$	$\Phi(c^-) \in \mathfrak{C}(c^-)$
$\frac{\mathbf{1}}{\boldsymbol{n}} < \frac{\boldsymbol{k}}{\boldsymbol{u}} \leq \frac{\mathbf{1}}{\boldsymbol{n-1}}$	$\Phi(c^-) \in \mathfrak{C}(c^-)$ if and only if $\frac{1}{n} < \frac{k}{u} \leq \frac{2n-2}{2n^2-3n}$
$\frac{\mathbf{1}}{\boldsymbol{n-1}} < \frac{\boldsymbol{k}}{\boldsymbol{u}} \leq \frac{\mathbf{1}}{\mathbf{2}}$	$\Phi(c^-) \notin \mathfrak{C}(c^-)$
$\frac{\mathbf{1}}{\mathbf{2}} < \frac{\boldsymbol{k}}{\boldsymbol{u}} \leq \mathbf{1}$	$\Phi(c^-) \notin \mathfrak{C}(c^-)$

Table 4: Summary of the findings in the four cases for $\lambda < 0$.

A summary of the findings in the previous four cases is given in Table 4, which can be put together as

$$
\Phi(c^-) \in \mathfrak{C}(c^-) \text{ if and only if } 0 < \frac{k}{u} \leq \frac{2n-2}{2n^2-3n}. \tag{45}
$$

Summary. As a final remark, we note that the numerical expressions we found under the assumption that $n \geq 3$ extend to the case of a two-players cooperation, despite the fact that the subcases in Tables 2 and 3 would not lead to the correct case. We observe that the summarizing conditions $\Phi(\lambda) \in \mathfrak{C}(\lambda)$ if and only if $\frac{(2n-1)(n-2)}{2n^2-3n} \leq \frac{k}{u} \leq 1$ for $\lambda > 0$, and $\Phi(\lambda) \in \mathfrak{C}(\lambda)$ if and only if $0 \leq \frac{k}{u} \leq \frac{2n-2}{2n^2-3n}$ for $\lambda < 0$ are valid for $n = 2$ as well. Indeed, the two terms $\frac{(2n-1)(n-2)}{2n^2-3n}$ and $\frac{2n-2}{2n^2-3n}$ equal 0 and 1, respectively, for $n = 2$, thus covering the stable case of a two-players cooperation.

Thanks to the previous observation and by combining the results of Cases 2 and 3, the proof is complete. □

E. Proof of Theorem 2 [Online]

To prove the theorem, we need the following Lemma.

Lemma 1. *Given a subadditive TU cost game (N, c) and a solution concept Ψ that is individually rational and efficient, let the individual rationality gap δ_i for company $i \in N$ be $\delta_i := c_i - \Psi_i$. Then, for any coalition $S \subsetneq N$, coalitional rationality implies bounded synergy for this coalition. Moreover, synergy greater than that of the grand-coalition leads to instability of the solution concept. In other words:*

$$\sum_{i \in S} \Psi_i \leq c_S \quad \Rightarrow \quad \sigma_S \leq \sigma_N \tag{46a}$$

$$\sigma_S > \sigma_N \quad \Rightarrow \quad \sum_{i \in S} \Psi_i > c_S \Rightarrow \mathfrak{C} = \emptyset \tag{46b}$$

Moreover,

$$\sum_{i \in S} \Psi_i = c_S + \varepsilon \quad \Leftrightarrow \quad \varepsilon = \sigma_S - \sigma_N + \sum_{i \in N \setminus S} \delta_i \tag{47}$$

Proof. We have:

$$\begin{aligned} & \sum_{i \in S} \Psi_i > c_S \\ \Leftrightarrow \quad & c_N - \sum_{i \in N \setminus S} \Psi_i + \sum_{i \in N} c_i - \sum_{i \in N} c_i > c_S \\ \Leftrightarrow \quad & \sigma_S > \sigma_N - \sum_{i \in N \setminus S} \delta_i \end{aligned}$$

Thus,

$$\sum_{i \in S} \Psi_i > c_S \quad \Leftrightarrow \quad \sigma_S > \sigma_N - \sum_{i \in N \setminus S} \delta_i \tag{48}$$

which implies (46a) and (46b). Equivalence (48) shows that instability of a coalition implies a synergy level higher than the difference of the synergy level of the grand-coalition and the individual rationality gap of the other players.

Equivalence (47) is obtained by repeating the same steps, but starting with $\sum_{i \in S} \Psi_i = c_S + \varepsilon$. □

Theorem 2. *Given a subadditive cost game (N,c) and an individually rational and efficient solution concept Ψ, the following holds:*

$$\sigma_S \leq \sigma_N \;\; \forall S \subseteq N \quad \Rightarrow \quad \Psi \in \mathfrak{C}_{\sigma_N}, \tag{49}$$

where $\mathfrak{C}_{\sigma_N}$ is the ε-Core for $\varepsilon = \sigma_N$.

Proof. Pick $S \subseteq N$ and assume that $\sigma_S \leq \sigma_N$. Then, either $\sigma_S \leq \sigma_N - \sum_{i\in N\setminus S}\delta_i$ or $\sigma_N - \sum_{i\in N\setminus S}\delta_i < \sigma_S \leq \sigma_N$. In the former case, we have $\sum_{i\in S}\Psi_i \leq c_S$; in the latter case, $\sum_{i\in S}\Psi_i > c_S$ and $\sum_{i\in S}\Psi_i = c_S + \varepsilon_S$, where $\varepsilon_S := \sigma_S - \sigma_N + \sum_{i\in N\setminus S}\delta_i$ (from (47)). From the condition $\sigma_S \leq \sigma_N$, we obtain that $\varepsilon_S \leq \varepsilon_0 := \sum_{i\in N}\delta_i$ for all coalitions $S \subseteq N$. If $\Psi \notin \mathfrak{C}$, then it follows that $\Psi \in \mathfrak{C}_\varepsilon$, where $\varepsilon = \max_{S\subseteq N}\{\sum_{i\in S}\Psi_i - c_S\} = \max_{S\subseteq N}\{\varepsilon_S\} \leq \varepsilon_0$. Thus, $\Psi \in \mathfrak{C}_{\varepsilon_0}$. If $\Psi \in \mathfrak{C}$, then $\Psi \in \mathfrak{C}_{\varepsilon_0}$ as $\mathfrak{C} \subseteq \mathfrak{C}_{\varepsilon_0}$.

Now, $\varepsilon_0 = \sum_{i\in N}(c_i - \Psi_i) = \sum_{i\in N} c_i - c_N = \sigma_N$ concludes the proof of (49). □

F. Proof of Corollary 2 [Online]

Corollary 2. *Given a parametric minimum cost flow game $c(\lambda)$, it follows that*

$$\Phi(\lambda) \in \mathfrak{C}_{\sigma_N(\lambda)} \quad \forall \lambda \in \Lambda. \tag{50}$$

Proof.

Consider a fixed value $\lambda \in \Lambda$. Non-emptiness of the core of the game $c(\lambda)$ implies that $\sigma_S(\lambda) \leq \sigma_N(\lambda)$ for all $S \subseteq N$. Indeed, non-emptiness of the core implies that there exists $x = (x_i)_{i\in N} \in \mathfrak{C}(\lambda)$ such that $\sum_{i\in N} x_i = c_N(\lambda)$ and $\sum_{i\in S} x_i \leq c_S(\lambda)$ for all coalitions $S \subseteq N$. From the inequalities $\sum_{i\in S} x_i \leq c_S(\lambda)$, we obtain

$$\sum_{i\in S} x_i \leq c_S(\lambda) \tag{51a}$$

$$\Leftrightarrow \quad c_N(\lambda) - \sum_{i\in N\setminus S} x_i + \sum_{i\in N} c_i(\lambda) - \sum_{i\in N} c_i(\lambda) \leq c_S(\lambda) \tag{51b}$$

$$\Leftrightarrow \quad \sigma_S(\lambda) \leq \sigma_N(\lambda) - \sum_{i\in N\setminus S}(c_i(\lambda) - x_i) \tag{51c}$$

$$\Rightarrow \sigma_S(\lambda) \leq \sigma_N(\lambda) \tag{51d}$$

where, in (51c), we note that $c_i(\lambda) - x_i \geq 0$ because of coalitional rationality. Thus, we have $\sigma_S(\lambda) \leq \sigma_N(\lambda)$ for all $S \subseteq N$. Therefore, since the Shapley value $\Phi(\lambda)$ is an individually rational and efficient solution concept, the claim follows from Theorem 2 and the fact that our argument holds true for each value $\lambda \in \Lambda$. □

G. Proofs of Theorems 3 and 4 [Online]

In this section, we provide the proofs of Theorems 3 and 4, which extend the results of Theorem 1 to a slightly richer network structure.

Independently of the position of the parameter λ on the arcs of the network, we can observe that the cost function $c^v(\lambda)$ of the vertical cooperation game $c^v(\lambda)$ can be decomposed into the sum of two games $c^v(\lambda) = c^h(\lambda) + \bar{c}(\lambda)$, where the horizontal cooperation game $c^h(\lambda)$ is defined by the minimum cost flow game from Section 4, where only the set of arcs $\{\bar{r}_i : i \in N\}$ is considered, and the cost function of the game $\bar{c}(\lambda)$ is the difference $\bar{c}(\lambda) \coloneqq c^v(\lambda) - c^h(\lambda)$. Note that, from the fact that the vertical cooperation path can be used only by the grand coalition, it follows that $\bar{c}_S(\lambda) \coloneqq 0$ for all $S \subsetneq N$, while only $\bar{c}_N(\lambda)$ can have non-zero value depending on λ and the parameters of the game.

Theorem 3. *Consider the vertical cooperation game $c^v(\lambda)$, where $c_{\bar{r}_{i_p}} = \bar{c} + \lambda$. For all values of direct unit transport cost c and capacity u, vertical unit transport cost $\bar{c}$ and capacity $\bar{u}$, amount of orders k and number of players n, we have*

$$\Phi^v(\lambda) \in \mathfrak{C}^v(\lambda) \qquad \text{for all } \lambda \in [-c, +\infty). \tag{52}$$

In other words, when the parametric cost is on one of the arcs in the vertical cooperation path, the Shapley value $\Phi^v(\lambda)$ is stable for all values of λ.

Proof. We first note that the game $c^v(\lambda)$ is symmetric for every value of λ. Indeed, for any two players $i, j \in N$, we have $c^v_{S\cup\{i\}}(\lambda) = c^v_{S\cup\{j\}}(\lambda)$ for all coalitions $S \subseteq N \setminus \{i,j\}$. This is because $S \cup \{i\} \subsetneq N$ and $S \cup \{j\} \subsetneq N$, so the vertical cooperation game reduces to the horizontal cooperation game where all players and their networks (being a single arc) are identical (see Figure 7). Therefore, we have that $c^v_{S\cup\{i\}}(\lambda) = c^h_{S\cup\{i\}}(\lambda) = c^h_{S\cup\{j\}}(\lambda) = c^v_{S\cup\{j\}}(\lambda)$. Note that this holds true even when $i = i_p$ as the parametric cost is on arc $\bar{r}_{i_p}$, which is not considered in the game $c^h(\lambda)$. It follows that $\Phi_i(\lambda) = \Phi_j(\lambda)$ for all i, j. Using efficiency of the Shapley value, this implies that $\Phi^v_i(\lambda) = \frac{1}{n} c^v_N(\lambda)$ for all players $i \in N$.

We now distinguish two cases depending on whether the unit transport cost $\lambda + n\bar{c}$ of the vertical cooperation path is greater than the unit cost c for direct transport or not. In the first case, the vertical cooperation path is not used by the cooperation, while in the second case, it might appear in the cost $c^v_N(\lambda)$ of the grand coalition.

Case 1: $\boldsymbol{\lambda + n\bar{c} > c}$. In this case, the game $c^v(\lambda)$ reduces to the game $c^h(0)$ as the vertical cooperation path is not used, and the parameter λ does not enter the cost expressions. Therefore, $c^h(\lambda) = c^h(0)$ and, thus, the Shapley value is stable by Theorem 1, i.e., $\Phi^v(\lambda) \in \mathfrak{C}^v(\lambda)$.

Case 2: $\boldsymbol{\lambda + n\bar{c} \leq c}$. In this case, the vertical cooperation path can be used by the grand coalition since its unit transport cost $\lambda + n\bar{c}$ is at most as large as the unit cost of direct transport. Testing coalitional rationality of the Shapley value requires us to test whether $\sum_{i\in S} \Phi_i^v(\lambda) \leq c_S^v(\lambda)$ for all coalitions $S \subseteq N$. For any $S \subseteq N$, we can reformulate the corresponding inequality as follows:

$$\sum_{i\in S} \Phi_i^v(\lambda) \leq c_S^v(\lambda) \tag{53a}$$

$$\Leftrightarrow \quad |S|\frac{c_N^v(\lambda)}{n} \leq |S|kc \tag{53b}$$

$$\Leftrightarrow \quad c_N^v(\lambda) \leq knc \tag{53c}$$

Inequality (53c) holds since $\lambda + n\bar{c} \leq c$ implies that $c_N^v(\lambda) = \bar{u}(\lambda + n\bar{c}) + (kn - \bar{u})c \leq knc$. Here, the equality follows since the grand coalition can transport $\bar{u}$ orders on the vertical cooperation path at unit cost $\lambda + n\bar{c}$. Therefore, coalitional rationality is satisfied for all coalitions $S \subseteq N$ and we obtain $\Phi^v(\lambda) \in \mathfrak{C}^v(\lambda)$. $\square$

We now provide the proof of Theorem 4.

Theorem 4. *In the vertical cooperation game $c^v(\lambda)$ where $c_{r_{ip}} = c + \lambda$, we have that for all values of direct unit transport cost c and capacity u, vertical unit transport cost $\bar{c}$ such that $n\bar{c} \leq c$ and capacity $\bar{u}$, amount of orders k and number of players n, the following holds: For each value of $\lambda \in [-\bar{c}, +\infty)$, stability of the Shapley value $\Phi^h(\lambda)$ in the horizontal cooperation game $c^h(\lambda)$ implies stability of the Shapley value $\Phi^v(\lambda)$ in the vertical cooperation game $c^v(\lambda)$. More formally:*

$$\text{For each } \lambda \in [-\bar{c}, +\infty): \qquad \Phi^h(\lambda) \in \mathfrak{C}^h(\lambda) \quad \Rightarrow \quad \Phi^v(\lambda) \in \mathfrak{C}^v(\lambda). \tag{54}$$

The converse does, in general, not hold true.

Proof. Note that this proof is based on observations found in the proof of Theorem 1 provided in Appendix D.

We first note that the game $\bar{c}(\lambda)$ is symmetric for every value of λ since only $\bar{c}_N(\lambda)$ can be non-zero. Therefore, for any two of players $i, j \in N$, we have $\bar{\Phi}_i(\lambda) = \bar{\Phi}_j(\lambda)$. Using efficiency of the Shapley value, this implies that $\bar{\Phi}_i(\lambda) = \frac{1}{n}\bar{c}_N(\lambda)$ for all players $i \in N$. Moreover, we note that $\bar{c}_N(\lambda) \leq 0$ for each λ. This follows since $\bar{c}_N(\lambda) = c_N^v(\lambda) - c_N^h(\lambda)$ by definition and since the graph considered in the minimum cost flow problem of the grand coalition N in $c^h(\lambda)$ is a subgraph of the graph in the minimum cost flow problem of the grand coalition in $c^v(\lambda)$, which implies that the minimum cost $c_N^h(\lambda)$ achievable by the grand coalition in $c^h(\lambda)$ is at least as large as the minimum cost $c_N^v(\lambda)$ achievable by the grand coalition in $c^v(\lambda)$.

Because the arc r_{i_p} with parametric cost is a direct transport arc, we obtain that the horizontal cooperation game $c^h(\lambda)$ can be decomposed as in the proof of Theorem 1: $c^h(\lambda) = c^0 + \lambda c^{\pm}$, where $c^{\pm} = c^+$ or $c^{\pm} = c^-$ depending on the sign of λ. The games c^0, c^+, and c^- have been defined in the proof of Theorem 1 provided in Appendix D (see Equations (7) and (27) for c^+ and c^-, respectively). Intuitively, the two games c^+ and c^- count only the amount of orders transported on arc r_{i_p}, so that $\lambda c^{\pm}$ is the cost difference resulting from transport on arc r_{i_p}. The dependence on the sign of λ follows from the fact that, for $\lambda < 0$, player i_p's arc r_{i_p} is used first in each coalition, while, for $\lambda > 0$, arc r_{i_p} is used last.

We let $\Phi^{\pm}$ and Φ^0 be the Shapley value for the games $c^{\pm}$, and c^0, respectively.

Combining the decomposition of $c^h(\lambda)$ with that of $c^v(\lambda)$, we obtain that $c^v(\lambda) = c^0 + \lambda c^{\pm} + \bar{c}(\lambda)$ and the Shapley value $\Phi^v(\lambda)$ can be computed by using linearity, as its expression is known explicitly for each of the games c^0, $c^{\pm}$ and $\bar{c}(\lambda)$.

Before entering the proof of coalitional rationality, we recall that $\bar{c}_S(\lambda) = 0$ for all $S \subsetneq N$.

Testing coalitional rationality for $S \subsetneq N$ means testing whether $\sum_{i \in S} \Phi_i^v(\lambda) \leq c_S^v(\lambda)$, which can be rewritten as follows:

$$\sum_{i \in S} \Phi_i^v(\lambda) \leq c_S^v(\lambda) \tag{55a}$$

$$\Leftrightarrow \quad \sum_{i \in S} \Phi_i^0 + \lambda \sum_{i \in S} \Phi_i^{\pm} + |S| \frac{\bar{c}_N(\lambda)}{n} \leq c_S^0 + \lambda c_S^{\pm} \tag{55b}$$

$$\Leftrightarrow \quad \lambda \sum_{i \in S} \Phi_i^{\pm} + |S| \frac{\bar{c}_N(\lambda)}{n} \leq \lambda c_S^{\pm} \tag{55c}$$

To further make the expression (55c) explicit, we distinguish two cases depending on the sign of λ.

Case 1: $\boldsymbol{\lambda = 0}$. In this case, (55c) reduces to

$$\frac{|S|}{n} \bar{c}_N(0) \leq 0, \tag{56}$$

which holds since $\bar{c}_N(0) \leq 0$. Thus, $\Phi^v(0) \in \mathfrak{C}^v(0)$.

Case 2: $\boldsymbol{\lambda > 0}$. In this case, we can rewrite (55c) as follows, knowing that $c^{\pm} = c^+$:

$$\sum_{i \in S} \Phi_i^+ + \frac{1}{\lambda} \frac{|S|}{n} \bar{c}_N(\lambda) \leq c_S^+ \tag{57}$$

Since $\bar{c}_N(\lambda) \leq 0$ and $\lambda > 0$, we have $\frac{1}{\lambda} \frac{|S|}{n} \bar{c}_N(\lambda) \leq 0$. Consequently, we obtain that $\sum_{i \in S} \Phi_i^+ \leq c_S^+$ for all $S \subset N$, which is equivalent to $\Phi^h(\lambda) \in \mathfrak{C}^h(\lambda)$ as seen in (8) in the proof of Theorem 1, implies (57), i.e., that $\Phi^v(\lambda) \in \mathfrak{C}^v(\lambda)$.

Case 3: $-c \leq \lambda < 0$. In this case, we rewrite (55c) as follows, knowing that $c^{\pm} = c^{-}$:

$$\sum_{i \in S} \Phi_i^- + \frac{1}{\lambda} \frac{|S|}{n} \bar{c}_N(\lambda) \geq c_S^- \tag{58}$$

Since $\bar{c}_N(\lambda) \leq 0$ and $\lambda < 0$, we have $\frac{1}{\lambda} \frac{|S|}{n} \bar{c}_N(\lambda) \geq 0$. Thus, if $\sum_{i \in S} \Phi_i^- \geq c_S^-$ for all $S \subseteq N$, then (58) holds and $\Phi^v(\lambda) \in \mathfrak{C}^v(\lambda)$. As $\sum_{i \in S} \Phi_i^- \geq c_S^-$ for all $S \subseteq N$ is equivalent to $\Phi^h(\lambda) \in \mathfrak{C}^h(\lambda)$ because of (28) in the proof of Theorem 1, we obtain that $\Phi^h(\lambda) \in \mathfrak{C}^h(\lambda)$ implies $\Phi^v(\lambda) \in \mathfrak{C}^v(\lambda)$ as claimed.

By combining the two cases, we have shown that $\Phi^h(\lambda) \in \mathfrak{C}^h(\lambda)$ implies $\Phi^v(\lambda) \in \mathfrak{C}^v(\lambda)$.

To show that the opposite implication does not hold true in general, we refer to the numerical example in Section 6.3.2, which shows that, for certain combinations of the parameters, we have $\Phi^h(\lambda) \in \mathfrak{C}^h(\lambda)$ but $\Phi^v(\lambda) \notin \mathfrak{C}^v(\lambda)$. □

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