Appendix A. Proof of Proposition 1

Consider a non-empty constrained queue set (i.e., $\Psi \neq \emptyset$) with a fixed N and n, such that the total number of permutations is $\binom{N}{n}$. Let "1" and "0" denote a CV and a non-CV, respectively. All of these permutations can be arranged into groups according to the location of the last CV, l_{cv}, in a constrained queue, where $l_{cv} \in [n, N]$. For each group of permutations, the SSDPRE can be applied to obtain the corresponding $\tilde{p}$. Table A enumerates all of the possible groups of permutations in which the last CV is located from the last place to the n^{th} place in the sequence.

Table A Enumeration of all of the possible groups of permutations under a constant N and n.

l_{cv}	Sample Permutation	$\tilde{p}$	Number of Permutations	Probability
N	$(\underbrace{0,0,1,1,0,\cdots,0,1,0,0}_{N-1},1)$	$S(n,N)$	$\binom{N-1}{n-1}$	$\binom{N-1}{n-1}/\binom{N}{n}$
$N-1$	$(\underbrace{0,0,1,1,0,\cdots,0,1,0}_{N-2},1,0)$	$S(n,N-1)$	$\binom{N-2}{n-1}$	$\binom{N-2}{n-1}/\binom{N}{n}$
$N-2$	$(\underbrace{0,0,1,0,\cdots,0,1,0}_{N-3},1,0,0)$	$S(n,N-2)$	$\binom{N-3}{n-1}$	$\binom{N-3}{n-1}/\binom{N}{n}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
n	$(\underbrace{1,1,1,\ldots,1,1}_{n-1},1,0,\cdots,0)$	$S(n,n)$	$\binom{n-1}{n-1}$	$\binom{n-1}{n-1}/\binom{N}{n}$

The mean of $\tilde{p}$, $E(\tilde{p})$, is the average of $\tilde{p}$ for all of the groups weighted by the probabilities, which automatically reduces to its definitional CV-to-total traffic ratio (i.e., n/N). Similarly, the variance of $\tilde{p}$ is the weighted variance. Detailed proofs follow.

(1) Proof of mean

If $n = 0$, $E(\tilde{p}) = 0 = \frac{n}{N}$.

If $n = 1$,

$$E(\tilde{p}) = \frac{\sum_{i=1}^{N}\binom{N-i}{0}S(1, N-i+1)}{\sum_{i=1}^{N}\binom{N-i}{0}} = \frac{\sum_{i=1}^{N-1} 0 + \sum_{i=N}^{N} 1}{N} = \frac{1}{N} = \frac{n}{N}. \quad \text{(A1)}$$

If $n > 1$,

$$E(\tilde{p}) = \frac{\sum_{i=1}^{N-n+1}\binom{N-i}{n-1}\frac{n-1}{N-i}}{\sum_{i=1}^{N-n+1}\binom{N-i}{n-1}}. \quad \text{(A2)}$$

As $\binom{N-i}{n-1}\frac{n-1}{N-i} = \binom{N-i-1}{n-2}$,

$$E(\tilde{p}) = \frac{\sum_{i=1}^{N-n+1}\binom{N-i-1}{n-2}}{\sum_{i=1}^{N-n+1}\binom{N-i}{n-1}}. \tag{A3}$$

According to **Appendixes B** and **C**, $\sum_{i=1}^{N-n+1}\binom{N-i}{n-1}=\binom{N}{n}$ and $\sum_{i=1}^{N-n+1}\binom{N-i-1}{n-2} = \binom{N-1}{n-1}$. Thus,

$$E(\tilde{p}) = \frac{\binom{N-1}{n-1}}{\binom{N}{n}} = \frac{n}{N}. \tag{A4}$$

(2) Proof of variance

If $n = 0$, $V(\tilde{p}) = 0$.

If $n = 1$,

$$\begin{aligned} Var(\tilde{p}) &= \frac{\sum_{i=1}^{N-n+1}\binom{N-i}{n-1}\big(S(n,N-i+1)-E(\tilde{p})\big)^2}{\sum_{i=1}^{N-n+1}\binom{N-i}{n-1}} \\ &= \frac{\sum_{i=1}^{N-1}(0-E(\tilde{p}))^2+\sum_{i=N}^{N}(1-E(\tilde{p}))^2}{N} = E(\tilde{p})^2 - \frac{2}{N}E(\tilde{p}) + \frac{1}{N}. \end{aligned} \tag{A5}$$

Substituting Eq. (A4) into Eq. (A5) gives:

$$Var(\tilde{p}) = \frac{n^2-2n+N}{N^2}. \tag{A6}$$

If $n > 1$,

$$Var(\tilde{p}) = \frac{\sum_{i=1}^{N-n+1}\binom{N-i}{n-1}\big(S(n,N-i+1)-E(\tilde{p})\big)^2}{\sum_{i=1}^{N-n+1}\binom{N-i}{n-1}} = \frac{\sum_{i=1}^{N-n+1}\binom{N-i}{n-1}(\frac{n-1}{N-i})^2}{\sum_{i=1}^{N-n+1}\binom{N-i}{n-1}} - E(\tilde{p})^2. \tag{A7}$$

Substituting $\sum_{i=1}^{N-n+1}\binom{N-i}{n-1} = \binom{N}{n}$ from **Appendix B**, $\binom{N-i}{n-1}\frac{n-1}{N-i} = \binom{N-i-1}{n-2}$, and Eq. (A4) into Eq. (A7) gives:

$$Var(\tilde{p}) = \frac{\sum_{i=1}^{N-n+1}\frac{n-1}{N-i}\binom{N-i-1}{n-2}}{\binom{N}{n}} - \frac{n^2}{N^2}. \tag{A8}$$

Q.E.D.

Appendix B. Derivation of $\sum_{i=1}^{N-n+1}\binom{N-i}{n-1}=\binom{N}{n}$

$$\sum_{i=1}^{N-n+1}\binom{N-i}{n-1} = \binom{N-1}{n-1} + \binom{N-2}{n-1} + \binom{N-3}{n-1} + \cdots + \binom{n+1}{n-1} + \binom{n}{n-1} + \binom{n-1}{n-1}. \tag{B1}$$

As $\binom{n-1}{n-1} = \binom{n}{n}$ and $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$,

$$\begin{aligned} \sum_{i=1}^{N-n+1}\binom{N-i}{n-1} &= \binom{N-1}{n-1} + \binom{N-2}{n-1} + \binom{N-3}{n-1} + \cdots + \binom{n+1}{n-1} + \binom{n}{n-1} + \binom{n}{n} \\ &= \binom{N-1}{n-1} + \binom{N-2}{n-1} + \binom{N-3}{n-1} + \cdots + \binom{n+1}{n-1} + \binom{n+1}{n} \end{aligned} \tag{B2}$$

…

$$= \binom{N}{n}.$$

Appendix C. Derivation of $\sum_{i=1}^{N-n+1}\binom{N-i-1}{n-2}$

$$\sum_{i=1}^{N-n+1}\binom{N-i-1}{n-2} = \binom{N-2}{n-2} + \binom{N-3}{n-2} + \cdots + \binom{n}{n-2} + \binom{n-1}{n-2} + \binom{n-2}{n-2}. \tag{C1}$$

As $\binom{n-2}{n-2} = \binom{n-1}{n-1}$ and $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$,

$$\begin{aligned} \sum_{i=1}^{N-n+1}\binom{N-i-1}{n-2} &= \binom{N-2}{n-2} + \binom{N-3}{n-2} + \cdots + \binom{n}{n-2} + \binom{n-1}{n-2} + \binom{n-1}{n-1} \\ &= \binom{N-2}{n-2} + \binom{N-2}{n-2} + \binom{N-3}{n-2} + \cdots + \binom{n}{n-2} + \binom{n}{n-1} \\ &\cdots \\ &= \binom{N-1}{n-1}. \end{aligned} \tag{C2}$$

Appendix D. Vertical queue experiments with Proposition 1

To validate and demonstrate the superiority of **Proposition 1**, a series of comprehensive simulation experiments based on the vertical queue assumption were conducted. Various combinations of fixed N and n were considered. The variability solely arose from the permutations of the stopped vehicles. Table D presents the simulation results. The means and variances for all of the cases based on **Proposition 1** were identical to the ground truths obtained via enumeration, demonstrating the accuracy of **Proposition 1**. Moreover, the negligible computation time of **Proposition 1** demonstrated its efficiency.

Table D Results of vertical queue experiments with Proposition 1.

N	n	Method	Mean	Variance	Computation time for mean (s)	Computation time for variance (s)
10	1	Ground truth	0.1	0.09000	0.001	0.002
		Proposition 1	0.1	0.09000	0	0
	3	Ground truth	0.3	0.01285	0.002	0.002
		Proposition 1	0.3	0.01285	0	0
	5	Ground truth	0.5	0.00668	0.001	0.002
		Proposition 1	0.5	0.00668	0	0
	7	Ground truth	0.7	0.00355	0.002	0.002
		Proposition 1	0.7	0.00355	0	0
	9	Ground truth	0.9	0.00111	0.001	0.001
		Proposition 1	0.9	0.00111	0	0
20	2	Ground truth	0.1	0.00867	0.002	0.002
		Proposition 1	0.1	0.00867	0	0
	6	Ground truth	0.3	0.00237	0.196	0.197
		Proposition 1	0.3	0.00237	0	0
	10	Ground truth	0.5	0.00145	0.977	0.978
		Proposition 1	0.5	0.00145	0	0

	14	Ground truth	0.7	0.00082	0.212	0.212
		Proposition 1	0.7	0.00082	0	0
	18	Ground truth	0.9	0.00026	0.001	0.001
		Proposition 1	0.9	0.00026	0	0
30	3	Ground truth	0.1	0.00233	0.025	0.025
		Proposition 1	0.1	0.00233	0	0
	9	Ground truth	0.3	0.00095	73.471	73.527
		Proposition 1	0.3	0.00095	0	0
	15	Ground truth	0.5	0.00061	943.923	944.490
		Proposition 1	0.5	0.00061	0	0
	21	Ground truth	0.7	0.00035	82.853	82.906
		Proposition 1	0.7	0.00035	0	0
	27	Ground truth	0.9	0.00012	0.028	0.028
		Proposition 1	0.9	0.00012	0	0

Appendix E. Proof of Proposition 2

Consider a non-empty constrained queue set (i.e., $\Psi \neq \emptyset$) with a fixed N and a varying n following $B(N,p)$, where n in each constrained queue varies and $n \in [0,N]$. It follows that the permutations of the constrained queues can be arranged into two levels. The first level contains all of the permutations grouped in terms of their value of n in the constrained queues. The second level contains a fixed N and n under each n, and thus **Proposition 1** can be employed. Similar to **Appendix A**, permutations in each group can be further arranged into sub-groups in terms of the location of the last CV, l_{cv}, in the constrained queues and the SSDPRE can be applied to each sub-group to obtain the corresponding $\tilde{p}$. Table E enumerates all of the possible groups and sub-groups of permutations.

Table E Enumeration of all of the possible permutations under a constant N and a varying n.

First level of n		Second level of l_{cv}	
n	Probability	$\tilde{p}$	Probability
0	$\binom{N}{0} p^0 (1-p)^N$	0	1
1	$\binom{N}{1} p^1 (1-p)^{N-1}$	$S(1, N-j+1), \forall j = 1,2,3,\dots,N$	$\binom{N-j}{0}/\binom{N}{1}, \forall j = 1,2,3,\dots,N$
2	$\binom{N}{2} p^2 (1-p)^{N-2}$	$S(2, N-j+1), \forall j = 1,2,3,\dots,N-1$	$\binom{N-j}{1}/\binom{N}{2}, \forall j = 1,2,3,\dots,N-1$
…	…	…	…
N	$\binom{N}{N} p^N (1-p)^0$	$S(N, N-j+1), \forall j = 1$	$\binom{N-j}{N-1}/\binom{N}{N}, \forall j = 1$

(1) Proof of mean

$$E(\tilde{p}) = 1 \cdot \binom{N}{0} p^0 (1-p)^N \cdot 0 + \sum_{i=1}^{N} \sum_{j=1}^{N-i+1} \left[\binom{N-j}{i-1}/\binom{N}{i}\right] S(i, N-j+1) \binom{N}{i} p^i (1-p)^{N-i}$$

$$= \sum_{i=1}^{1} \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} p^i (1-p)^{N-i} S(i, N-j+1)$$
$$+ \sum_{i=2}^{N} \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} p^i (1-p)^{N-i} S(i, N-j+1). \tag{E1}$$

Let $\sum_{i=1}^{1} \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} p^i (1-p)^{N-i} S(i, N-j+1)$ and $\sum_{i=2}^{N} \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} p^i (1-p)^{N-i} S(i, N-j+1)$ be A and B, respectively. According to **Appendix F**, $A = p(1-p)^{N-1}$ and $B = p - p(1-p)^{N-1}$. Therefore,

$$E(\tilde{p}) = A + B = p(1-p)^{N-1} + p - p(1-p)^{N-1} = p = n/N. \tag{E2}$$

(2) Proof of variance

$$Var(\tilde{p}) = \binom{N}{0} p^0 (1-p)^N \cdot 1[0 - E(\tilde{p})]^2$$
$$+ \sum_{i=1}^{N} \sum_{j=1}^{N-i+1} \frac{\binom{N-j}{i-1}}{\binom{N}{i}} \binom{N}{i} p^i (1-p)^{N-i} [S(i, N-j+1) - E(\tilde{p})]^2$$
$$= (1-p)^N E(\tilde{p})^2 + \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} p^i (1-p)^{N-i} [S(i, N-j+1) - E(\tilde{p})]^2 \tag{E3}$$
$$+ \sum_{i=2}^{N} \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} p^i (1-p)^{N-i} [S(i, N-j+1) - E(\tilde{p})]^2.$$

Let $C = \sum_{i=1}^{1} \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} p^i (1-p)^{N-i} [S(i, N-j+1) - E(\tilde{p})]^2$ and $D = \sum_{i=2}^{N} \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} p^i (1-p)^{N-i} [S(i, N-j+1) - E(\tilde{p})]^2$, respectively. Thus,

$$Var(\tilde{p}) = (1-p)^N E(\tilde{p})^2 + C \qquad if\ N = 1, \tag{E4}$$

$$Var(\tilde{p}) = (1-p)^N E(\tilde{p})^2 + C + D \quad if\ N > 1. \tag{E5}$$

According to **Appendix G**, $C = p(1-p)^{N-1}[NE(\tilde{p})^2 - 2E(\tilde{p}) + 1]$ and $D = \sum_{i=2}^{N} p^i (1-p)^{N-i} \sum_{j=1}^{N-i+1} \binom{N-j-1}{i-2} \frac{i-1}{N-j} - 2E(\tilde{p}) p [1 - (1-p)^{N-1}] + E(\tilde{p})^2 [1 - (1-p)^N - Np(1-p)^{N-1}]$. By substituting C, D, Eq. (3b), and Eq. (E2) into Eq. (E4) and Eq. (E5),

$$Var(\tilde{p}) = p(1-p) \quad if\ N = 1, \tag{E6}$$

$$Var(\tilde{p}) = \sum_{i=2}^{N} p^i (1-p)^{N-i} \left[V_1(i, N) + (\tfrac{i}{N})^2 \right] \binom{N}{i} - p^2 + p(1-p)^{N-1} \quad if\ N > 1. \tag{E7}$$

Q.E.D.

Appendix F. Derivations of A **and** B

For A,

$$A = \sum_{i=1}^{1} \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} p^i (1-p)^{N-i} S(i, N-j+1). \tag{F1}$$

$$= \sum_{j=1}^{N} \binom{N-j}{0} p^1 (1-p)^{N-1} S(1, N-j+1)$$

$$= \sum_{j=1}^{N-1} \binom{N-j}{0} p^1 (1-p)^{-1} \cdot 0 + \sum_{j=N}^{N} \binom{N-j}{0} p^1 (1-p)^{N-1} \cdot 1$$

$$= 0 + p^1 (1-p)^{N-1} = p^1 (1-p)^{N-1}.$$

For B,

$$\begin{aligned} B &= \sum_{i=2}^{N} \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} p^i (1-p)^{N-i} S(i, N-j+1) \\ &= \sum_{i=2}^{N} \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} \frac{i-1}{N-j} p^i (1-p)^{N-i}. \end{aligned} \tag{F2}$$

Given that $\binom{N-j}{i-1} \frac{i-1}{N-j} = \binom{N-j-1}{i-2}$, Eq. (F2) can be rewritten as

$$B = \sum_{i=2}^{N} \sum_{j=1}^{N-i+1} \binom{N-j-1}{i-2} p^i (1-p)^{N-i}. \tag{F3}$$

As

$$\begin{aligned} \sum_{j=1}^{N-i+1} \binom{N-j-1}{i-2} &= \binom{N-2}{i-2} + \binom{N-3}{i-2} + \binom{N-4}{i-2} + \cdots + \binom{i-1}{i-2} + \binom{i-2}{i-2} \\ &= \binom{N-2}{i-2} + \binom{N-3}{i-2} + \binom{N-4}{i-2} + \cdots + \binom{i-1}{i-2} + \binom{i-1}{i-1} \\ &= \binom{N-2}{i-2} + \binom{N-3}{i-2} + \binom{N-4}{i-2} + \cdots + \binom{i}{i-1} \\ &\cdots \\ &= \binom{N-1}{i-1}, \end{aligned} \tag{F4}$$

and thus substituting Eq. (F4) into Eq. (F3) affords

$$B = \sum_{i=2}^{N} \binom{N-1}{i-1} p^i (1-p)^{N-i}. \tag{F5}$$

As $\binom{N-1}{i-1} = \binom{N}{i} \frac{i}{N}$,

$$\begin{aligned} B &= \frac{1}{N} \sum_{i=2}^{N} i \binom{N}{i} p^i (1-p)^{N-i} \\ &= \frac{1}{N} \left[\sum_{i=0}^{N} i \binom{N}{i} p^i (1-p)^{N-i} - 0 - \left(1 \binom{N}{1} p^1 (1-p)^{N-1} \right) \right] \\ &= \frac{1}{N} \left[\left(E(n) - Np(1-p)^{N-1} \right) \right] = p - p(1-p)^{N-1}. \end{aligned} \tag{F6}$$

Appendix G. Derivations of *C* and *D*

For C,

$$C = \sum_{i=1}^{1} \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} p^i (1-p)^{N-i} [S(i, N-j+1) - E(\tilde{p})]^2 \tag{G1}$$

$$= p(1-p)^{N-1} [N E(\tilde{p})^2 - 2E(\tilde{p}) + 1].$$

For D,

$$D = \sum_{i=2}^{N} \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} p^i (1-p)^{N-i} [S(i, N-j+1) - E(\tilde{p})]^2$$
$$= \sum_{i=2}^{N} p^i (1-p)^{N-i} \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} \left[\frac{i-1}{N-j} - E(\tilde{p})\right]^2. \tag{G2}$$

Let $F = \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} \left[\frac{i-1}{N-j} - E(\tilde{p})\right]^2$, thus,

$$F = \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} (\frac{i-1}{N-j})^2 - 2E(\tilde{p}) \sum_{j=1}^{N-i+1} \binom{N-j}{i-1} \frac{i-1}{N-j} + E(\tilde{p})^2 \sum_{j=1}^{N-i+1} \binom{N-j}{i-1}. \tag{G3}$$

As $\binom{N-j}{i-1} \frac{i-1}{N-j} = \binom{N-j-1}{i-2}$,

$$F = \sum_{j=1}^{N-i+1} \binom{N-j-1}{i-2} \frac{i-1}{N-j} - 2E(\tilde{p}) \sum_{j=1}^{N-i+1} \binom{N-j-1}{i-2} + E(\tilde{p})^2 \sum_{j=1}^{N-i+1} \binom{N-j}{i-1}. \tag{G4}$$

Therefore, by substituting Eq. (B2) and Eq. (F4) into Eq. (G4), it is found that

$$F = \sum_{j=1}^{N-i+1} \binom{N-j-1}{i-2} \frac{i-1}{N-j} - 2E(\tilde{p}) \binom{N-1}{i-1} + E(\tilde{p})^2 \binom{N}{i}. \tag{G5}$$

Thus, D can be rewritten as

$$D = \sum_{i=2}^{N} p^i (1-p)^{N-i} F = \sum_{i=2}^{N} p^i (1-p)^{N-i} \sum_{j=1}^{N-i+1} \binom{N-j-1}{i-2} \frac{i-1}{N-j}$$
$$- \sum_{i=2}^{N} p^i (1-p)^{N-i} 2E(\tilde{p}) \binom{N-1}{i-1} + \sum_{i=2}^{N} p^i (1-p)^{N-i} E(\tilde{p})^2 \binom{N}{i} . \tag{G6}$$

Let $G = \sum_{i=2}^{N} p^i (1-p)^{N-i} 2E(\tilde{p}) \binom{N-1}{i-1}$ and $H = \sum_{i=2}^{N} p^i (1-p)^{N-i} E(\tilde{p})^2 \binom{N}{i}$. According to Eqs. (F5) and (F6),

$$G = \sum_{i=2}^{N} p^i (1-p)^{N-i} 2E(\tilde{p}) \binom{N-1}{i-1} = 2E(\tilde{p}) p [1 - (1-p)^{N-1}]. \tag{G7}$$

For H,

$$H = \sum_{i=2}^{N} p^i (1-p)^{N-i} E(\tilde{p})^2 \binom{N}{i}$$
$$= E(\tilde{p})^2 [1 - (1-p)^N - Np(1-p)^{N-1}]. \tag{G8}$$

Thus,

$$D = \sum_{i=2}^{N} p^i (1-p)^{N-i} \sum_{j=1}^{N-i+1} \binom{N-j-1}{i-2} \frac{i-1}{N-j} - 2E(\tilde{p}) p [1 - (1-p)^{N-1}]$$
$$+ E(\tilde{p})^2 [1 - (1-p)^N - Np(1-p)^{N-1}]. \tag{G9}$$

Appendix H. Vertical queue experiments using Proposition 2

To validate and demonstrate the superiority of **Proposition 2**, a series of comprehensive simulation experiments based on the vertical queue assumption were conducted. Various combinations of fixed N and varying $n \sim B(N, p)$ were considered. In this context, variability originated from the permutations of the stopped vehicles and n. First, for each chosen N, all of the possible n were enumerated. Then, under each given n, both N and n were fixed and the problem was reduced to the conditions stated in **Proposition 1**. Thus, all of the permutations could be enumerated and the corresponding $\tilde{p}$ were evaluated using the SSDPRE. Table H reports the simulation results. The means and variances of all of the cases based on **Proposition 2** were identical to the ground truths obtained via enumeration, demonstrating the accuracy of **Proposition 2**. Moreover, the negligible computation time demonstrated the efficiency of **Proposition 2**.

Table H Results of vertical queue experiments using Proposition 2.

N	p	Method	Mean	Variance	Computation time for mean (s)	Computation time for variance (s)
10	0.1	Ground truth	0.1	0.04915	0.013	0.013
		Proposition 2	0.1	0.04915	0	0
	0.3	Ground truth	0.3	0.04337	0.011	0.011
		Proposition 2	0.3	0.04337	0	0
	0.5	Ground truth	0.5	0.03342	0.011	0.012
		Proposition 2	0.5	0.03342	0	0
	0.7	Ground truth	0.7	0.02479	0.012	0.012
		Proposition 2	0.7	0.02479	0	0
	0.9	Ground truth	0.9	0.01014	0.012	0.012
		Proposition 2	0.9	0.01014	0	0
20	0.1	Ground truth	0.1	0.02110	11.052	11.131
		Proposition 2	0.1	0.02110	0	0.001
	0.3	Ground truth	0.3	0.01351	11.126	11.210
		Proposition 2	0.3	0.01351	0	0.001
	0.5	Ground truth	0.5	0.01400	11.082	11.170
		Proposition 2	0.5	0.01400	0	0.001
	0.7	Ground truth	0.7	0.01133	10.968	11.051
		Proposition 2	0.7	0.01133	0	0.001
	0.9	Ground truth	0.9	0.00477	11.046	11.133
		Proposition 2	0.9	0.00477	0	0.001
30	0.1	Ground truth	0.1	0.00970	11160.706	11617.295
		Proposition 2	0.1	0.00970	0	0.002
	0.3	Ground truth	0.3	0.00801	11163.901	11618.435
		Proposition 2	0.3	0.00801	0	0.002
	0.5	Ground truth	0.5	0.00895	11161.346	11617.299
		Proposition 2	0.5	0.00895	0	0.002
	0.7	Ground truth	0.7	0.00736	11160.903	11616.992
		Proposition 2	0.7	0.00736	0	0.002
	0.9	Ground truth	0.9	0.00312	11162.348	11618.472
		Proposition 2	0.9	0.00312	0	0.002

Appendix I. Proof of Corollary 2

The detailed proof of the joint probability distribution of observing any combination of n and $\widetilde{N}$ is provided in this appendix. If $n = 0$, $\widetilde{N}$ must be 0 and N can be either $0, k, \forall k \in \mathbb{N}^+$. When $N = 0$, the probability for this case is π_0. When $N = k, \forall k \in \mathbb{N}^+$, the probability of observing k vehicle is π_k and the probability of observing k non-CVs is $(1-p)^k$. Thus, the probability for this case is $\pi_k(1-p)^k$. Therefore, the probability of observing $n = 0$ and $\widetilde{N} = 0$ is given by Eq. (I1):

$$P\left(n = 0, \widetilde{N} = 0\right) = \pi_0 + \sum_{z=1}^{k} \pi_z \, (1-p)^z. \tag{I1}$$

If $n > 0$, $\widetilde{N} > 0$. $\forall i, j = 1, 2, \dots, k,$ where $j \geq i$,

$$P\left(n = i, \widetilde{N} = j\right) = \sum_{z=j}^{k} \pi_z \binom{j-1}{i-1} p^i (1-p)^{z-i}. \tag{I2}$$

Therefore, the joint probability distribution of observing any combination of n and $\widetilde{N}$ is given by Eq. (G3):

$$P\left(n = i, \widetilde{N} = j\right) = \begin{cases} \pi_0 + \sum_{z=1}^{k} \pi_z \, (1-p)^z, & i = 0, j = 0 \\ \sum_{z=j}^{k} \pi_z \binom{j-1}{i-1} p^i (1-p)^{z-i}, & \forall i, j = 1, 2, \dots, k, j \geq i \end{cases}, \tag{I3}$$

where $\pi_i = P(N = i), \forall i \in \mathbb{N}$.

Q.E.D.

Appendix J. Vertical queue experiments using Proposition 3

Comprehensive simulation experiments under the vertical queue assumption were conducted to examine the performance of **Proposition 3**. By assuming N follows a Poisson distribution, $Pois(\lambda)$, **Corollary 1** was obtained. Various combinations of λ and p were considered. In this case, variability stemmed from N, n, and the permutation of the stopped vehicles. As $N \sim Pois(\lambda)$, there were infinitely many possible outcomes for N and it was not possible to enumerate all possible permutations. Thus, to estimate the mean and variance of $\tilde{p}$, random sampling of the possible permutations based on the selected $Pois(\lambda)$ and $B(N, p)$ with three selected sample sizes (i.e., 100,000, 1,000,000 and 10,000,000) was conducted, which provided useful references for the validation of **Corollary 1**. Various ranges of constrained queue length k were considered for the validation based on **Corollary 1**. Table J presents the results of the experiments on **Corollary 1**. The results show that estimates based on **Corollary 1** (or **Proposition 3**) were more accurate and computationally efficient than those obtained by random sampling.

Table J Results of vertical queue experiments using Corollary 1.

λ	p	Method	k/# of samples	Mean	Variance	Computing time for mean (s)	Computing time for variance (s)
10	0.1	Sampling	100,000	0.100	0.04870	3.539	3.546
			1,000,000	0.100	0.04875	35.243	35.303
			10,000,000	0.100	0.04892	350.512	351.087
		Corollary 1	[0, 20]	0.1	0.05068	0	0.010
			[0, 30]	0.1	0.05071	0	0.025
			[0, 40]	0.1	0.05071	0	0.054
	0.3	Sampling	100,000	0.300	0.05057	3.653	3.658
			1,000,000	0.300	0.05069	36.786	36.829
			10,000,000	0.300	0.05073	367.545	367.976
		Corollary 1	[0, 20]	0.3	0.05304	0	0.010
			[0, 30]	0.3	0.05306	0	0.025
			[0, 40]	0.3	0.05306	0	0.053
	0.5	Sampling	100,000	0.500	0.04128	3.680	3.683
			1,000,000	0.500	0.04127	37.309	37.347
			10,000,000	0.500	0.04129	369.133	369.505
		Corollary 1	[0, 20]	0.5	0.04277	0	0.009
			[0, 30]	0.5	0.04279	0	0.026
			[0, 40]	0.5	0.04279	0	0.055
	0.7	Sampling	100,000	0.701	0.02949	3.692	3.696
			1,000,000	0.700	0.02957	37.303	37.340
			10,000,000	0.700	0.02945	374.259	374.627
		Corollary 1	[0, 20]	0.7	0.03041	0	0.010
			[0, 30]	0.7	0.03042	0	0.026
			[0, 40]	0.7	0.03042	0	0.054
	0.9	Sampling	100,000	0.900	0.01169	3.681	3.684
			1,000,000	0.900	0.01173	37.823	37.858
			10,000,000	0.900	0.01172	377.504	377.864
		Corollary 1	[0, 20]	0.9	0.01207	0	0.010
			[0, 30]	0.9	0.01208	0	0.027
			[0, 40]	0.9	0.01208	0	0.054
20	0.1	Sampling	100,000	0.099	0.02153	4.243	4.248
			1,000,000	0.100	0.02202	43.884	43.937
			10,000,000	0.100	0.02205	430.435	430.954
		Corollary 1	[10, 30]	0.1	0.02229	0	0.024
			[0, 40]	0.1	0.02268	0	0.053
			[0, 50]	0.1	0.02268	0	0.098
	0.3	Sampling	100,000	0.300	0.01527	4.350	4.353
			1,000,000	0.300	0.01505	43.430	43.465
			10,000,000	0.300	0.01502	442.631	443.001
		Corollary 1	[10, 30]	0.3	0.01496	0	0.024
			[0, 40]	0.3	0.01536	0	0.054
			[0, 50]	0.3	0.01536	0	0.098
	0.5	Sampling	100,000	0.499	0.01475	4.331	4.334
			1,000,000	0.500	0.01478	43.794	43.831

			10,000,000	0.500	0.01481	434.439	434.803
		Corollary 1	[10, 30]	0.5	0.01477	0	0.025
			[0, 40]	0.5	0.01511	0	0.055
			[0, 50]	0.5	0.01511	0	0.098
	0.7	Sampling	100,000	0.700	0.01190	4.365	4.368
			1,000,000	0.700	0.01185	44.013	44.050
			10,000,000	0.700	0.01186	438.506	438.864
		Corollary 1	[10, 30]	0.7	0.01185	0	0.024
			[0, 40]	0.7	0.01210	0	0.055
			[0, 50]	0.7	0.01210	0	0.099
	0.9	Sampling	100,000	0.900	0.00502	4.339	4.343
			1,000,000	0.900	0.00499	43.706	43.743
			10,000,000	0.900	0.00497	432.398	432.579
		Corollary 1	[10, 30]	0.9	0.00497	0	0.024
			[0, 40]	0.9	0.00507	0	0.054
			[0, 50]	0.9	0.00507	0	0.099
30	0.1	Sampling	100,000	0.100	0.01062	4.793	4.798
			1,000,000	0.100	0.01042	47.707	47.752
			10,000,000	0.100	0.01041	482.325	482.749
		Corollary 1	[20, 40]	0.1	0.00997	0	0.046
			[10, 50]	0.1	0.01067	0	0.098
			[0, 60]	0.1	0.01067	0	0.174
	0.3	Sampling	100,000	0.300	0.00834	4.804	4.808
			1,000,000	0.300	0.00834	48.285	48.323
			10,000,000	0.300	0.00834	483.857	484.219
		Corollary 1	[20, 40]	0.3	0.00793	0	0.046
			[10, 50]	0.3	0.00846	0	0.097
			[0, 60]	0.3	0.00847	0	0.163
	0.5	Sampling	100,000	0.500	0.00920	4.814	4.818
			1,000,000	0.500	0.00920	48.994	49.031
			10,000,000	0.500	0.00921	487.016	487.387
		Corollary 1	[20, 40]	0.5	0.00878	0	0.045
			[10, 50]	0.5	0.00933	0	0.096
			[0, 60]	0.5	0.00934	0	0.162
	0.7	Sampling	100,000	0.700	0.00753	4.826	4.830
			1,000,000	0.700	0.00752	48.148	48.183
			10,000,000	0.700	0.00755	486.355	486.721
		Corollary 1	[20, 40]	0.7	0.00720	0	0.047
			[10, 50]	0.7	0.00765	0	0.098
			[0, 60]	0.7	0.00765	0	0.164
	0.9	Sampling	100,000	0.900	0.00321	4.784	4.788
			1,000,000	0.900	0.00320	48.222	48.258
			10,000,000	0.900	0.00320	485.871	486.233
		Corollary 1	[20, 40]	0.9	0.00305	0	0.046
			[10, 50]	0.9	0.00324	0	0.097
			[0, 60]	0.9	0.00324	0	0.163

Appendix K. Computation costs of Propositions 4 and 5

Comprehensive simulation experiments based on the vertical queue assumption were conducted to compare the computation efficiency of **Propositions 4** and **5**. Various combinations of red periods, V/C ratios, and ranges of constrained queue length k (i.e., $k \in [0, 10]$, $k \in [0, 13]$, and $k \in [0, 15]$)

were considered. The constrained queue length distributions obtained based on **Propositions 4** and **5** were identical. However, the results clearly demonstrated that **Proposition 5** is much more efficient than **Proposition 4**.

Table K Results of vertical queue experiments using **Propositions 4** and **5.**

No.	r	V/C	k	Proposition 4 (s)	Proposition 5 (s)
1	15	0.3	[0,10]	0.351	0.027
			[0,13]	50.614	0.062
			[0,15]	2045.923	0.113
2	15	0.5	[0,10]	0.349	0.026
			[0,13]	50.867	0.063
			[0,15]	2046.345	0.113
3	15	0.7	[0,10]	0.353	0.028
			[0,13]	51.284	0.063
			[0,15]	2047.389	0.116
4	30	0.3	[0,10]	0.350	0.027
			[0,13]	50.589	0.065
			[0,15]	2045.940	0.113
5	30	0.5	[0,10]	0.353	0.028
			[0,13]	50.682	0.064
			[0,15]	2045.356	0.112
6	30	0.7	[0,10]	0.349	0.027
			[0,13]	50.489	0.063
			[0,15]	2045.334	0.118
7	45	0.3	[0,10]	0.349	0.027
			[0,13]	50.540	0.064
			[0,15]	2045.457	0.101
8	45	0.5	[0,10]	0.351	0.028
			[0,13]	50.998	0.062
			[0,15]	2046.312	0.112
9	45	0.7	[0,10]	0.350	0.031
			[0,13]	50.234	0.064
			[0,15]	2045.465	0.112

Appendix L. Calibration of PDT model based on CTL model

Let $\xi(q, r, k\tau)$ be the abstract form of the PDT model with parameters of traffic demand q, red period r, and dissipation time $k\tau$. When an actual constrained queue set is considered, the PDT model is given by $\xi(q, r', \mu k\tau)$, where $r' = r - \Delta$ and $\mu = 1 - \frac{\Delta}{r}$. Let $N_{PDT}(\Delta)$ be the expectation of an actual constrained queue length based on the PDT model, i.e., $E[\xi(q, r', \mu k\tau)]$. The objective function for the calibration of the net loss of red time Δ is thus given by Eq. (L1), as follows:

$$\min_{\Delta} Z = \sum[N_{PDT}(\Delta) - N_{obs}]^2, \tag{L1}$$

where N_{obs} is the observed average constrained queue length. Equation (L1) searches for an optimal Δ $s.t.$ the sum of squared differences between the expected actual constrained queue length and the observed constrained queue length. The optimal Δ can be obtained via a simple line search.

Based on the simulation data presented in Section 5.4, the optimal Δ was found to be 5.048 s. Thus, the calibrated PDT model based on the CTL model was $\xi\left[(q, r-5.048, (1-\frac{5.048}{r})k\tau)\right]$.

Appendix M. Calibration of the CDT model based on the CTL model

Let $N_{CDT}(\Delta)$ be the average actual constrained queue length based on the CDT model. Then, from Eq. (31),

$$N_{CDT}(\Delta) = \frac{sq(r-\Delta)}{s-q}. \tag{M1}$$

Similar to **Appendix L**, an objective function for the calibration of the net loss of red time Δ is formulated as the minimization of the sum of squared differences between the average actual constrained queue length and the observed constrained queue length, as follows:

$$\min_{\Delta} Z = \sum[N_{CDT}(\Delta) - N_{obs}]^2, \tag{M2}$$

By setting $\partial Z/\partial\Delta = 0$,

$$\frac{\partial Z}{\partial \Delta} = -\sum 2\left(\frac{sq(r-\Delta)}{s-q} - N_{obs}\right)\frac{sq}{s-q} = 0. \tag{M3}$$

It follows that

$$\Delta = \frac{\sum\frac{sq}{s-q}\left(\frac{sqr}{s-q} - N_{obs}\right)}{\sum\left(\frac{sq}{s-q}\right)^2}. \tag{M4}$$

Based on the simulation data presented in Section 5.4, the optimal Δ was found to be 7.270 s. Thus, the calibrated CDT model based on the CTL model was $N_{CDT} = \frac{sq(r-7.270)}{s-q}$.

Appendix N. Estimations of real-time delays

This appendix provides a method of estimating delays in real time for two approaches joining at an intersection controlled by a simplified red–green–amber signal structure. To simplify the delay estimation, it was assumed that vehicle arrivals followed uniform distributions and no residual vehicles carried over. Using simple geometry, the predicted delays in cycle i+1 for approaches 1 and 2, $D_{i+1,1}$ and $D_{i+1,2}$, were estimated as

$$D_{i+1,1} = \begin{cases} \dfrac{s\tilde{q}_{i+1,1}{r_{i+1,1}}^2}{2(s-\tilde{q}_{i+1,1})} & if\ \tilde{q}_{i+1,1} \le \dfrac{sg_{i+1,1}}{C} \\ \dfrac{\tilde{q}_{i+1,1}{r_{i+1,1}}^2 + g_{i+1,1}\left[\tilde{q}_{i+1,1}\left(2C-g_{i+1,1}\right) - sg_{i+1,1}\right]}{2} & if\ \tilde{q}_{i+1,1} > \dfrac{sg_{i+1,1}}{C} \end{cases}, \quad \text{(N1)}$$

$$D_{i+1,2} = \begin{cases} \dfrac{1}{2}(2L_{i+1,2} + \tilde{q}_{i+1,2}r_0)r_0 + \dfrac{(\tilde{q}_{i+1,2}r_{i+1,2})^2}{2(s-\tilde{q}_{i+1,2})} + \dfrac{1}{2}\tilde{q}_{i+1,2}\left(r_{i+1,2}-r_0\right)^2 & if\ \tilde{q}_{i+1,2} \le \dfrac{sg_{i+1,2}-L_{i+1,2}}{r_0+g_{i+1,2}} \\ \dfrac{1}{2}(2L_{i+1,2} + \tilde{q}_{i+1,2}C)C - \dfrac{1}{2}(2C - g_{i+1,2} - 2r_0)sg_{i+1,2} & if\ \tilde{q}_{i+1,2} > \dfrac{sg_{i+1,2}-L_{i+1,2}}{r_0+g_{i+1,2}} \end{cases}, \quad \text{(N2)}$$

where $r_{i+1,j}$ and $g_{i+1,j}$ are respectively the effective red and effective green in cycle $i+1$ on approach j; $\tilde{q}_{i+1,j}$ is the predicted real-time average arrival rate in cycle $i+1$ on approach j and $\tilde{q}_{i+1,j} = M_{i,j}/C$; s is the saturation flow rate; C is the cycle length; r_0 is the clearance loss time comprising a portion of the amber period and the all-red clearance time (which was 4 seconds in this case); and $L_{i+1,2} = \tilde{q}_{i+1,2}(r_{i,2} - r_0)$.