Appendix A: Proofs of Theorems and Lemmas

A.1. Proof of Theorem 1

Proof. The length δ of time slice for discretization is small enough and the model is approximate to the corresponding continuous-time bottleneck model (Arnott, de Palma, and Lindsey 1990a). Thus, there exist time slices i_1, i_2, and i_3 so that the UE solution $\mathbf{r}^*$ satisfies

$$r_i^* = 0, C_i(\mathbf{r}^*) \geq \mu, \text{for } i = 1, 2, \cdots, i_1 - 1, i_3 + 1, \cdots, M, \tag{A.1}$$

$$r_i^* > 0, C_i(\mathbf{r}^*) = \mu, \text{for } i = i_1, i_1 + 1, \cdots, i_3, \tag{A.2}$$

$$C_i(\mathbf{r}^*) = \begin{cases} (\alpha - \beta)\left(T^f + \frac{1}{s}\sum_{j=i_1}^{i}(r_j^* - s)\delta\right) + \beta(t^* - i\delta), \text{ for } i = i_1, i_1 + 1, \cdots, i_2, \\ (\alpha + \gamma)\left(T^f + \frac{1}{s}\sum_{j=i_1}^{i}(r_j^* - s)\delta\right) + \gamma(i\delta - t^*), \text{ for } i = i_2 + 1, i_2 + 2, \cdots, i_3, \end{cases} \tag{A.3}$$

where μ is the minimum travel cost of commuters at the UE solution. Expression (A.3) is obtained by formulae (3) to (6). Expressions (A.1) to (A.3) mean that all commuters enter the network in a rush hour from time slice i_1 to i_3 at the UE solution. Commuters who enter the network in the interval from time slice i_1 to i_2 arrive at the destination earlier than t^* or on time, and commuters who enter the network in the interval from time slice i_2+1 to i_3 arrive at the destination later than t^*. There is always queue at the bottleneck from time slice i_1 to i_3. Without loss of generality, it is assumed that

$$C_{i_1 - 1}(\mathbf{r}^*) = \mu_1 > \mu. \tag{A.4}$$

The flow rate vector $\mathbf{r}^0$ is taken as

$$r_i^0 = \begin{cases} 0, & \text{for } i = 1, 2, \cdots, i_1 - 2, i_3 + 1, \cdots, M, \\ \hat{r}, & \text{for } i = i_1 - 1, \\ r_i^* + \tilde{r}, & \text{for } i = i_1, i_1 + 1, \cdots, i_1 + i_4 - 1, \\ r_i^*, & \text{for } i = i_1 + i_4, \cdots, i_3 - i_4 - 1, \\ r_i^* - \tilde{r}, & \text{for } i = i_3 - i_4, \cdots, i_3 - 1, \\ r_i^* - \hat{r}, & \text{for } i = i_3, \end{cases} \tag{A.5}$$

where the constants $\tilde{r}$ and $\hat{r}$ and the integer i_4 satisfy

$$0 < \tilde{r} \leq r_i^*, \text{for } i = i_3 - i_4, \cdots, i_3 - 1, \tag{A.6}$$

$$0 < \hat{r} \leq r_i^* \leq s, \text{for } i = i_3, \tag{A.7}$$

$$\hat{r} \leq \frac{\tilde{r}^2 \delta i_4}{4s(\mu_1 - \mu)}, \tag{A.8}$$

$$\frac{2\alpha + \gamma - \beta + 1}{\beta + \gamma} \leq i_4 \leq \min\{i_2 - i_1 + 1, i_3 - i_2 - 1\}. \tag{A.9}$$

The integer i_4 is the number of time slices for flow transfer from the late arrival interval to the early arrival interval and it is adopted to give the equilibrium flow rate vector $\mathbf{r}^*$ a perturbation. Obviously, $\mathbf{r}^0$ belongs to a neighborhood $N(\mathbf{r}^*) \subset \Omega$ of the UE point. Given the equilibrium flow rate vector $\mathbf{r}^*$, $\mathbf{r}^0$ is generated by transferring flow rate $\tilde{r}$ at each time slice $i = i_3 - i_4, \cdots, i_3 - 1$ to each time slice from i_1 to $i_1 + i_4 - 1$ respectively and transferring flow rate $\hat{r}$ at time slice i_3 to time slice $i_1 - 1$. After these flow transfers,

commuters who enter the network at time slice i_1 to i_1+i_4-1 arrive at the destination earlier than t^* or on time, commuters who enter the network at time slice i_3-i_4 to i_3-1 arrive at the destination later than t^*, and there is always queue at the bottleneck from time slice i_1 to i_3. Therefore, by formulae (3) to (6) and (A.1) to (A.5), the travel costs of commuters entering the network at time slice i_1-1 to i_1+i_4-1 and i_3-i_4 to i_3 are formulated as

$$C_i(\mathbf{r}^0)=\mu_1, \text{for } i=i_1-1, \tag{A.10}$$

$$C_i(\mathbf{r}^0)=(\alpha-\beta)\left(T^f+\tfrac{1}{s}\sum_{j=i_1}^{i}(r_j^*+\tilde{r}-s)\delta\right)+\beta(t^*-i\delta)=\mu+\tfrac{(\alpha-\beta)\tilde{r}\delta(i-i_1+1)}{s}, \quad \text{for } i=i_1,i_1+1,\cdots,i_1+i_4-1, \tag{A.11}$$

$$C_i(\mathbf{r}^0)=(\alpha+\gamma)\left(T^f+\frac{1}{s}\left(\sum_{j=i_1}^{i_1+i_4-1}(r_j^*+\tilde{r}-s)\delta+\sum_{j=i_1+i_4}^{i_3-i_4-1}(r_j^*-s)\delta+\sum_{j=i_3-i_4}^{i}(r_j^*-\tilde{r}-s)\delta\right)\right)$$
$$+\gamma(i\delta-t^*)=\mu+\frac{(\alpha+\gamma)\tilde{r}\delta(i_3-i-1)}{s}, \text{for } i=i_3-i_4,\cdots,i_3-1. \tag{A.12}$$

$$C_i(\mathbf{r}^0)=\mu, \text{for } i=i_3, \tag{A.13}$$

Thus, we have

$$(\mathbf{r}^0-\mathbf{r}^*)^{\mathrm{T}}\mathbf{C}(\mathbf{r}^*)=\hat{r}\mu_1+\sum_{i=i_1}^{i_1+i_4-1}\tilde{r}\mu-\sum_{i=i_3-i_4}^{i_3-1}\tilde{r}\mu-\hat{r}\mu=\hat{r}(\mu_1-\mu)>0, \tag{A.14}$$

$$\begin{aligned}(\mathbf{r}^0-\mathbf{r}^*)^{\mathrm{T}}\mathbf{C}(\mathbf{r}^0)&=\hat{r}\mu_1+\sum_{i=i_1}^{i_1+i_4-1}\tilde{r}\left(\mu+\frac{(\alpha-\beta)\tilde{r}\delta(i-i_1+1)}{s}\right)-\sum_{i=i_3-i_4}^{i_3-1}\tilde{r}\left(\mu+\frac{(\alpha+\gamma)\tilde{r}\delta(i_3-i-1)}{s}\right)-\hat{r}\mu\\&=\hat{r}\mu_1+i_4\tilde{r}\mu+\frac{(\alpha-\beta)\tilde{r}^2\delta}{s}\frac{(i_4+1)i_4}{2}-i_4\tilde{r}\mu-\frac{(\alpha+\gamma)\tilde{r}^2\delta}{s}\frac{(i_4-1)i_4}{2}-\hat{r}\mu\\&=\hat{r}(\mu_1-\mu)+\frac{(2\alpha+\gamma-\beta-(\beta+\gamma)i_4)\,\tilde{r}^2\delta i_4}{2s}\le\frac{\tilde{r}^2\delta i_4}{4s}-\frac{\tilde{r}^2\delta i_4}{2s}=-\frac{\tilde{r}^2\delta i_4}{4s}<0.\end{aligned} \tag{A.15}$$

In equation (A.14), the first equality is generated by substituting expressions (A.1), (A.2), (A.4), and (A.5) into its left hand side, and the inequality follows the condition $\mu_1>\mu$. In equation (A.15), the first equality is obtained by substituting formulae (A.1), (A.5), and (A.10) to (A.13) into its left hand side, the first inequality follows relations (A.8) and (A.9), and the second inequality is obtained by the conditions $\tilde{r}>0$, $\delta>0$, $i_4>0$, and $s>0$.

Inequalities (A.14) and (A.15) indicate that the travel cost vector function $\mathbf{C}(\cdot)$ is non-quasi monotone with respect to $\mathbf{r}$ on $N(\mathbf{r}^*)$. This completes the proof.

A.2. Proof of Theorem 2

Proof. The necessity is first proved. If $\tilde{\mathbf{r}}^{(m+1)}=\tilde{\mathbf{r}}^{(m)}$, then

$$\begin{aligned}&\sum\nolimits_{k=1}^{K}\sum\nolimits_{i=1}^{M}\left(\tilde{r}_{k,i}^{(m+1)}-\tilde{r}_{k,i}^{(m)}\right)C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)\\&=\tilde{\lambda}^{(m)}\left(\sum\nolimits_{k=1}^{K}\sum\nolimits_{i=1}^{i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)}\sum\nolimits_{l=1}^{K}\sum\nolimits_{j=1}^{i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)}\left(G_{lj,ki}\left(\tilde{\mathbf{r}}^{(m)}\right)-G_{ki,lj}\left(\tilde{\mathbf{r}}^{(m)}\right)\right)C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)\right.\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{K}\sum_{i=i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)+1}^{M}\sum_{l=1}^{K}\sum_{j=i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)+1}^{M}\left(G_{lj,ki}\left(\tilde{\mathbf{r}}^{(m)}\right)-G_{ki,lj}\left(\tilde{\mathbf{r}}^{(m)}\right)\right)C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)\Bigg)\\
=&\,\tilde{\lambda}^{(m)}\Bigg(\sum_{l=1}^{K}\sum_{j=1}^{i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)}\sum_{k=1}^{K}\sum_{i=1}^{i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)}G_{ki,lj}\left(\tilde{\mathbf{r}}^{(m)}\right)C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right)\\
&-\sum_{k=1}^{K}\sum_{i=1}^{i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)}\sum_{l=1}^{K}\sum_{j=1}^{i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)}G_{ki,lj}\left(\tilde{\mathbf{r}}^{(m)}\right)C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)\\
&+\sum_{l=1}^{K}\sum_{j=i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)+1}^{M}\sum_{k=1}^{K}\sum_{i=i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)+1}^{M}G_{ki,lj}\left(\tilde{\mathbf{r}}^{(m)}\right)C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right)\\
&-\sum_{k=1}^{K}\sum_{i=i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)+1}^{M}\sum_{l=1}^{K}\sum_{j=i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)+1}^{M}G_{ki,lj}\left(\tilde{\mathbf{r}}^{(m)}\right)C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)\Bigg)\\
=&-\tilde{\lambda}^{(m)}\Bigg(\sum_{k=1}^{K}\sum_{i=1}^{i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)}\sum_{l=1}^{K}\sum_{j=1}^{i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)}G_{ki,lj}\left(\tilde{\mathbf{r}}^{(m)}\right)\left(C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)-C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right)\right)\\
&+\sum_{k=1}^{K}\sum_{i=i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)+1}^{M}\sum_{l=1}^{K}\sum_{j=i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)+1}^{M}G_{ki,lj}\left(\tilde{\mathbf{r}}^{(m)}\right)\left(C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)-C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right)\right)\Bigg)\Bigg)=0. \qquad \text{(A.16)}
\end{aligned}
$$

In equation (A.16), the first equality is generated by substituting formulae (10) and (11) into its left hand side. The second equality is obtained by the relationships

$$
\begin{aligned}
&\sum_{k=1}^{K}\sum_{i=1}^{i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)}\sum_{l=1}^{K}\sum_{j=1}^{i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)}G_{lj,ki}\left(\tilde{\mathbf{r}}^{(m)}\right)C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)\\
&\quad=\sum_{l=1}^{K}\sum_{j=1}^{i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)}\sum_{k=1}^{K}\sum_{i=1}^{i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)}G_{ki,lj}\left(\tilde{\mathbf{r}}^{(m)}\right)C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right)\ \text{and}\\
&\sum_{k=1}^{K}\sum_{i=i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)+1}^{M}\sum_{l=1}^{K}\sum_{j=i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)+1}^{M}G_{lj,ki}\left(\tilde{\mathbf{r}}^{(m)}\right)C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)\\
&\quad=\sum_{l=1}^{K}\sum_{j=i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)+1}^{M}\sum_{k=1}^{K}\sum_{i=i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)+1}^{M}G_{ki,lj}\left(\tilde{\mathbf{r}}^{(m)}\right)C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right).
\end{aligned}
$$

Each term $G_{ki,lj}\left(\tilde{\mathbf{r}}^{(m)}\right)\left(C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)-C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right)\right)$ on the left hand side of the last equality in equation (A.16) is non-negative. Therefore, it follows from the last equality in equation (A.16) and formula (9) that

$$
\begin{aligned}
&G_{ki,lj}\left(\tilde{\mathbf{r}}^{(m)}\right)\left(C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)-C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right)\right)=\tilde{r}_{k,i}^{(m)}\left[C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)-C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right)\right]_+^2=0,\\
&\qquad\text{for } k=1,2,\cdots,K, i=1,2,\cdots,i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right), l=1,2,\cdots,K, \text{and } j=1,2,\cdots,i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right), \qquad \text{(A.17)}\\
&G_{ki,lj}\left(\tilde{\mathbf{r}}^{(m)}\right)\left(C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)-C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right)\right)=\tilde{r}_{k,i}^{(m)}\left[C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)-C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right)\right]_+^2=0,\\
&\qquad\text{for } k=1,2,\cdots,K, i=i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)+1,\cdots,M, l=1,2,\cdots,K, \text{and } j=i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)+1,\cdots,M. \qquad \text{(A.18)}
\end{aligned}
$$

Namely, $\tilde{\mathbf{r}}^{(m)}$ satisfies conditions (16) and (17).

The sufficiency is then proved. If $\tilde{\mathbf{r}}^{(m)}$ satisfies conditions (16) and (17), then expressions (A.17) and (A.18) hold. By formulae (10) and (11), it is generated that $\tilde{\mathbf{r}}^{(m+1)}=\tilde{\mathbf{r}}^{(m)}$ holds. This completes the proof.

A.3. Proof of Lemma 1

Proof. The time slice length δ is a positive constant, hence it is removed from the gap function $V(\cdot)$ in this proof. By formula (22), the partial derivative of the function $V_1(\cdot)$ with respect to $r_{\bar{k},\bar{i}}$ is formulated as

$$
\begin{aligned}
\frac{\partial V_1(\mathbf{r})}{\partial r_{\bar{k},\bar{i}}}=&\sum_{l=1}^{K}\sum_{j=1}^{i_l(\mathbf{r}_l)}\left[C_{\bar{k},\bar{i}}(\mathbf{r}_{\bar{k}})-C_{l,j}(\mathbf{r}_l)\right]_+^2+2\sum_{k=1}^{K}\sum_{i=1}^{i_k(\mathbf{r}_k)}\sum_{l=1}^{K}\sum_{j=1}^{i_l(\mathbf{r}_l)}G_{ki,lj}(\mathbf{r})\left(\frac{\partial C_{k,i}(\mathbf{r}_k)}{\partial r_{\bar{k},\bar{i}}}-\frac{\partial C_{l,j}(\mathbf{r}_l)}{\partial r_{\bar{k},\bar{i}}}\right)\\
=&\sum_{l=1}^{K}\sum_{j=1}^{i_l(\mathbf{r}_l)}\left[C_{\bar{k},\bar{i}}(\mathbf{r}_{\bar{k}})-C_{l,j}(\mathbf{r}_l)\right]_+^2-2\sum_{k=1}^{K}\sum_{i=1}^{i_k(\mathbf{r}_k)}\tilde{H}_{k,i}(\mathbf{r})\frac{\partial C_{k,i}(\mathbf{r}_k)}{\partial r_{\bar{k},\bar{i}}},\\
&\qquad\text{for } \bar{k}=1,2,\cdots,K \text{ and } \bar{i}=1,2,\cdots,i_{\bar{k}}(\mathbf{r}_{\bar{k}}); \qquad \text{(A.19)}
\end{aligned}
$$

$$\frac{\partial V_1(\mathbf{r})}{\partial r_{\bar{k},\bar{i}}} = 2\sum_{k=1}^{K}\sum_{i=1}^{i_k(\mathbf{r}_k)}\sum_{l=1}^{K}\sum_{j=1}^{i_l(\mathbf{r}_l)} G_{ki,lj}(\mathbf{r})\left(\frac{\partial C_{k,i}(\mathbf{r}_k)}{\partial r_{\bar{k},\bar{i}}} - \frac{\partial C_{l,j}(\mathbf{r}_l)}{\partial r_{\bar{k},\bar{i}}}\right) = -2\sum_{k=1}^{K}\sum_{i=1}^{i_k(\mathbf{r}_k)} \tilde{H}_{k,i}(\mathbf{r})\frac{\partial C_{k,i}(\mathbf{r}_k)}{\partial r_{\bar{k},\bar{i}}},$$
$$\text{for } \bar{k}=1,2,\cdots,K \text{ and } \bar{i}=i_{\bar{k}}(\mathbf{r}_{\bar{k}})+1,\cdots,M. \tag{A.20}$$

In formulae (A.19) and (A.20), $G_{ki,lj}(\mathbf{r})$ and $\tilde{U}_{k,i}(\mathbf{r})$ are expressed as formulae (9) and (11) respectively.

Without loss of generality, it is assumed that there is queue at bottlenecks on the first k_0 ($\leq K$) links in iteration m during the second phase of the sub-procedure in Step 2. By formulae (19) and (20), the partial derivatives of the travel cost vector function are then governed by

$$\frac{\partial C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)}{\partial \tilde{r}_{\bar{k},\bar{i}}^{(m)}} = \begin{cases} \frac{(\alpha-\beta)\delta}{s_k}, \text{ for } k=1,\cdots,k_0,\ i=i_{k,1},\cdots,i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right),\ \bar{k}=k, \text{ and } \bar{i}=i_{k,1},\cdots,i, \\ \frac{(\alpha+\gamma)\delta}{s_k}, \text{ for } k=1,\cdots,k_0,\ i=i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)+1,\cdots,i_{k,2},\ \bar{k}=k, \text{ and } \bar{i}=i_{k,1},\cdots,i, \\ 0, \qquad \text{otherwise.} \end{cases} \tag{A.21}$$

In iteration m during the second phase of the sub-procedure in Step 2, the inner product between the update direction of flow rates and the gradient of the function $V_1(\cdot)$ is deduced as

$$\begin{aligned}
&\sum_{\bar{k}=1}^{K}\sum_{\bar{i}=1}^{M} \tilde{H}_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}^{(m)}\right)\frac{\partial V_1\left(\tilde{\mathbf{r}}^{(m)}\right)}{\partial \tilde{r}_{\bar{k},\bar{i}}^{(m)}} \\
&\quad= \sum_{\bar{k}=1}^{K}\sum_{\bar{i}=1}^{i_{\bar{k}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)}\sum_{\bar{l}=1}^{K}\sum_{\bar{j}=1}^{i_{\bar{l}}\left(\tilde{\mathbf{r}}_{\bar{l}}^{(m)}\right)}\sum_{l=1}^{K}\sum_{j=1}^{i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)} \left(G_{\bar{l}\bar{j},\bar{k}\bar{i}}\left(\tilde{\mathbf{r}}^{(m)}\right) - G_{\bar{k}\bar{i},\bar{l}\bar{j}}\left(\tilde{\mathbf{r}}^{(m)}\right)\right)\left[C_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right) - C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right)\right]_+^2 \\
&\qquad -2\sum_{\bar{k}=1}^{K}\sum_{\bar{i}=1}^{i_{\bar{k}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)}\sum_{k=1}^{K}\sum_{i=1}^{i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)} \tilde{H}_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}^{(m)}\right)\tilde{H}_{k,i}\left(\tilde{\mathbf{r}}^{(m)}\right)\frac{\partial C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)}{\partial \tilde{r}_{\bar{k},\bar{i}}^{(m)}} \\
&\qquad -2\sum_{\bar{k}=1}^{K}\sum_{\bar{i}=i_{\bar{k}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)+1}^{M}\sum_{k=1}^{K}\sum_{i=1}^{i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)} \tilde{H}_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}^{(m)}\right)\tilde{H}_{k,i}\left(\tilde{\mathbf{r}}^{(m)}\right)\frac{\partial C_{k,i}\left(\tilde{\mathbf{r}}_k^{(m)}\right)}{\partial \tilde{r}_{\bar{k},\bar{i}}^{(m)}} \\
&\quad= -\sum_{\bar{k}=1}^{K}\sum_{\bar{i}=1}^{i_{\bar{k}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)}\sum_{\bar{l}=1}^{K}\sum_{\bar{j}=1}^{i_{\bar{l}}\left(\tilde{\mathbf{r}}_{\bar{l}}^{(m)}\right)}\sum_{l=1}^{K}\sum_{j=1}^{i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)} G_{\bar{k}\bar{i},\bar{l}\bar{j}}\left(\tilde{\mathbf{r}}^{(m)}\right) \\
&\qquad \times\left(\left[C_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right) - C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right)\right]_+^2 - \left[C_{\bar{l},\bar{j}}\left(\tilde{\mathbf{r}}_{\bar{l}}^{(m)}\right) - C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right)\right]_+^2\right) \\
&\qquad -2\sum_{k=1}^{k_0}\sum_{i=i_{k,1}}^{i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)}\sum_{\bar{i}=i_{k,1}}^{i} \tilde{H}_{k,i}\left(\tilde{\mathbf{r}}^{(m)}\right)\tilde{H}_{k,\bar{i}}\left(\tilde{\mathbf{r}}^{(m)}\right)\frac{(\alpha-\beta)\delta}{s_k}.
\end{aligned} \tag{A.22}$$

In equation (A.22), the first equality is generated by introducing formulae (11), (A.19), and (A.20) into its left hand side. The second equality is obtained by introducing formula (A.21) into its left hand side.

By expression (9), the first term on the right hand side of equation (A.22) can be further deduced as

$$\begin{aligned}
&-\sum_{\bar{k}=1}^{K}\sum_{\bar{i}=1}^{i_{\bar{k}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)}\sum_{\bar{l}=1}^{K}\sum_{\bar{j}=1}^{i_{\bar{l}}\left(\tilde{\mathbf{r}}_{\bar{l}}^{(m)}\right)}\sum_{l=1}^{K}\sum_{j=1}^{i_l\left(\tilde{\mathbf{r}}_l^{(m)}\right)} G_{\bar{k}\bar{i},\bar{l}\bar{j}}\left(\tilde{\mathbf{r}}^{(m)}\right)\left(\left[C_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right) - C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right)\right]_+^2 - \left[C_{\bar{l},\bar{j}}\left(\tilde{\mathbf{r}}_{\bar{l}}^{(m)}\right) - C_{l,j}\left(\tilde{\mathbf{r}}_l^{(m)}\right)\right]_+^2\right) \\
&\quad\leq -\sum_{\bar{k}=1}^{K}\sum_{\bar{i}=1}^{i_{\bar{k}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)}\sum_{\bar{l}=1}^{K}\sum_{\bar{j}=1}^{i_{\bar{l}}\left(\tilde{\mathbf{r}}_{\bar{l}}^{(m)}\right)} \tilde{r}_{\bar{k},\bar{i}}^{(m)}\left[C_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right) - C_{\bar{l},\bar{j}}\left(\tilde{\mathbf{r}}_{\bar{l}}^{(m)}\right)\right]_+^3,
\end{aligned} \tag{A.23}$$

and the second term can be further deduced as

$$-2\sum_{k=1}^{k_0}\sum_{i=i_{k,1}}^{i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)}\sum_{\bar{i}=i_{k,1}}^{i}\tilde{H}_{k,i}\left(\tilde{\mathbf{r}}^{(m)}\right)\tilde{H}_{k,\bar{i}}\left(\tilde{\mathbf{r}}^{(m)}\right)\frac{(\alpha-\beta)\delta}{s_k}$$
$$=-\sum_{k=1}^{k_0}\frac{(\alpha-\beta)\delta}{s_k}\left(\left(\sum\nolimits_{i=i_{k,1}}^{i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)}\tilde{H}_{k,i}\left(\tilde{\mathbf{r}}^{(m)}\right)\right)^2+\sum\nolimits_{i=i_{k,1}}^{i_k\left(\tilde{\mathbf{r}}_k^{(m)}\right)}\tilde{H}_{k,i}\left(\tilde{\mathbf{r}}^{(m)}\right)^2\right). \tag{A.24}$$

If the distribution of flow rates in the early arrival intervals is at non-equilibrium state in iteration m, then both the terms on the right hand side of equation (A.22) are negative. Thus, the inner product between the update direction of flow rates and the gradient of the function $V_1(\cdot)$ is negative, i.e., inequality (24) holds. This completes the proof.

A.4. Proof of Lemma 2

Proof. The derivative of V_1 with respect to $\tilde{\lambda}$ is written as

$$dV_1(\tilde{\lambda})\Big/d\tilde{\lambda}=\nabla V_1\left(\tilde{\mathbf{r}}^{(m)}+\tilde{\lambda}\tilde{\mathbf{H}}\left(\tilde{\mathbf{r}}^{(m)}\right)\right)^{\mathrm{T}}\tilde{\mathbf{H}}\left(\tilde{\mathbf{r}}^{(m)}\right).$$

By Lemma 1, $dV_1(\tilde{\lambda})\Big/d\tilde{\lambda}<0$ for $\tilde{\lambda}=0$ in iteration m at non-equilibrium state. Meanwhile, $dV_1\Big/d\tilde{\lambda}$ is continuous with respect to $\tilde{\lambda}\in\left[0,\hat{\lambda}^{(m)}\right]$. Therefore, there exists $\tilde{\lambda}_0^{(m)}\in\left(0,\hat{\lambda}^{(m)}\right]$ such that

$$dV_1(\tilde{\lambda})\Big/d\tilde{\lambda}<0,\text{for }\tilde{\lambda}\in\left[0,\tilde{\lambda}_0^{(m)}\right),\text{and }dV_1(\tilde{\lambda})\Big/d\tilde{\lambda}=0,\text{for }\tilde{\lambda}=\tilde{\lambda}_0^{(m)},$$

or

$$dV_1(\tilde{\lambda})\Big/d\tilde{\lambda}<0,\text{for }\tilde{\lambda}\in\left[0,\hat{\lambda}^{(m)}\right],\text{and }\tilde{\lambda}_0^{(m)}=\hat{\lambda}^{(m)}.$$

As a result, the value of the function $V_1(\cdot)$ will decrease when set $\tilde{\lambda}^{(m)}=\tilde{\lambda}_0^{(m)}$ in each iteration m at non-equilibrium state until its value becomes zero. $V_1(\mathbf{r})=0$ holds if and only if $\mathbf{r}\in\Omega$ satisfies condition (16). Therefore, the flow rates in the early arrival intervals will converge to UE. This completes the proof.

A.5. Proof of Lemma 3

Proof. The time slice length δ is a positive constant, hence it is removed from the gap function $V(\cdot)$ in this proof. By formula (23), the partial derivative of the function $V_2(\cdot)$ with respect to $r_{\bar{k},\bar{i}}$ is formulated as

$$\frac{\partial V_2(\mathbf{r})}{\partial r_{\bar{k},\bar{i}}}=2\sum_{k=1}^{K}\sum_{i=i_k(\mathbf{r}_k)+1}^{M}\sum_{l=1}^{K}\sum_{j=i_l(\mathbf{r}_l)+1}^{M}G_{ki,lj}(\mathbf{r})\left(\frac{\partial C_{k,i}(\mathbf{r}_k)}{\partial r_{\bar{k},\bar{i}}}-\frac{\partial C_{l,j}(\mathbf{r}_l)}{\partial r_{\bar{k},\bar{i}}}\right)=-2\sum_{k=1}^{K}\sum_{i=i_k(\mathbf{r}_k)+1}^{M}\tilde{H}_{k,i}(\mathbf{r})\frac{\partial C_{k,i}(\mathbf{r}_k)}{\partial r_{\bar{k},\bar{i}}},$$
$$\text{for }\bar{k}=1,2,\cdots,K\text{ and }\bar{i}=1,2,\cdots,i_{\bar{k}}(\mathbf{r}_{\bar{k}}); \tag{A.25}$$

$$\frac{\partial V_2(\mathbf{r})}{\partial r_{\bar{k},\bar{i}}}=\sum_{l=1}^{K}\sum_{j=i_l(\mathbf{r}_l)+1}^{M}\left[C_{\bar{k},\bar{i}}(\mathbf{r}_{\bar{k}})-C_{l,j}(\mathbf{r}_l)\right]_+^2+2\sum_{k=1}^{K}\sum_{i=i_k(\mathbf{r}_k)+1}^{M}\sum_{l=1}^{K}\sum_{j=i_l(\mathbf{r}_l)+1}^{M}G_{ki,lj}(\mathbf{r})\left(\frac{\partial C_{k,i}(\mathbf{r}_k)}{\partial r_{\bar{k},\bar{i}}}-\frac{\partial C_{l,j}(\mathbf{r}_l)}{\partial r_{\bar{k},\bar{i}}}\right)$$
$$=\sum_{l=1}^{K}\sum_{j=i_l(\mathbf{r}_l)+1}^{M}\left[C_{\bar{k},\bar{i}}(\mathbf{r}_{\bar{k}})-C_{l,j}(\mathbf{r}_l)\right]_+^2-2\sum_{k=1}^{K}\sum_{i=i_k(\mathbf{r}_k)+1}^{M}\tilde{H}_{k,i}(\mathbf{r})\frac{\partial C_{k,i}(\mathbf{r}_k)}{\partial r_{\bar{k},\bar{i}}},$$
$$\text{for }\bar{k}=1,2,\cdots,K\text{ and }\bar{i}=i_{\bar{k}}(\mathbf{r}_{\bar{k}})+1,\cdots,M. \tag{A.26}$$

In formulae (A.25) and (A.26), $G_{ki,lj}(\mathbf{r})$ and $\tilde{H}_{k,i}(\mathbf{r})$ are expressed as formulae (9) and (11).

Without loss of generality, it is assumed that there is queue at bottlenecks on the first k_0 ($\leq K$) links in iteration m during the third phase of the sub-procedure in Step 2. The partial derivatives of the travel cost

vector function are then governed by expression (A.21). At the same time, the distribution of flow rates in the early arrival intervals is already at the UE state during the third phase.

In iteration m during the third phase of the sub-procedure in Step 2, the inner product between the update direction of flow rates and the gradient of the function $V(\cdot)$ is deduced as

$$\begin{aligned}
&\sum_{\bar{k}=1}^{K}\sum_{\bar{i}=1}^{M}\tilde{H}_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}^{(m)}\right)\frac{\partial V\left(\tilde{\mathbf{r}}^{(m)}\right)}{\partial \tilde{r}_{\bar{k},\bar{i}}^{(m)}}=\sum_{\bar{k}=1}^{K}\sum_{\bar{i}=i_{\bar{k}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)+1}^{M}\tilde{H}_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}^{(m)}\right)\frac{\partial V\left(\tilde{\mathbf{r}}^{(m)}\right)}{\partial \tilde{r}_{\bar{k},\bar{i}}^{(m)}}\\
&=-2\sum_{\bar{k}=1}^{K}\sum_{\bar{i}=i_{\bar{k}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)+1}^{M}\sum_{k=1}^{K}\sum_{i=1}^{i_k\left(\tilde{\mathbf{r}}_{k}^{(m)}\right)}\tilde{H}_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}^{(m)}\right)\tilde{H}_{k,i}\left(\tilde{\mathbf{r}}^{(m)}\right)\frac{\partial C_{k,i}\left(\tilde{\mathbf{r}}_{k}^{(m)}\right)}{\partial \tilde{r}_{\bar{k},\bar{i}}^{(m)}}\\
&+\sum_{\bar{k}=1}^{K}\sum_{\bar{i}=i_{\bar{k}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)+1}^{M}\sum_{\bar{l}=1}^{K}\sum_{\bar{j}=i_{\bar{l}}\left(\tilde{\mathbf{r}}_{\bar{l}}^{(m)}\right)+1}^{M}\sum_{l=1}^{K}\sum_{j=i_l\left(\tilde{\mathbf{r}}_{l}^{(m)}\right)+1}^{M}\left(G_{\bar{l}\bar{j},\bar{k}\bar{i}}\left(\tilde{\mathbf{r}}^{(m)}\right)-G_{\bar{k}\bar{i},\bar{l}\bar{j}}\left(\tilde{\mathbf{r}}^{(m)}\right)\right)\left[C_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)-C_{l,j}\left(\tilde{\mathbf{r}}_{l}^{(m)}\right)\right]_{+}^{2}\\
&-2\sum_{\bar{k}=1}^{K}\sum_{\bar{i}=i_{\bar{k}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)+1}^{M}\sum_{k=1}^{K}\sum_{i=i_k\left(\tilde{\mathbf{r}}_{k}^{(m)}\right)+1}^{M}\tilde{H}_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}^{(m)}\right)\tilde{H}_{k,i}\left(\tilde{\mathbf{r}}^{(m)}\right)\frac{\partial C_{k,i}\left(\tilde{\mathbf{r}}_{k}^{(m)}\right)}{\partial \tilde{r}_{\bar{k},\bar{i}}^{(m)}}\\
&=-\sum_{\bar{k}=1}^{K}\sum_{\bar{i}=i_{\bar{k}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)+1}^{M}\sum_{\bar{l}=1}^{K}\sum_{\bar{j}=i_{\bar{l}}\left(\tilde{\mathbf{r}}_{\bar{l}}^{(m)}\right)+1}^{M}\sum_{l=1}^{K}\sum_{j=i_l\left(\tilde{\mathbf{r}}_{l}^{(m)}\right)+1}^{M}G_{\bar{k}\bar{i},\bar{l}\bar{j}}\left(\tilde{\mathbf{r}}^{(m)}\right)\\
&\times\left(\left[C_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)-C_{l,j}\left(\tilde{\mathbf{r}}_{l}^{(m)}\right)\right]_{+}^{2}-\left[C_{\bar{l},\bar{j}}\left(\tilde{\mathbf{r}}_{\bar{l}}^{(m)}\right)-C_{l,j}\left(\tilde{\mathbf{r}}_{l}^{(m)}\right)\right]_{+}^{2}\right)\\
&-2\sum_{k=1}^{k_0}\sum_{i=i_k\left(\tilde{\mathbf{r}}_{k}^{(m)}\right)+1}^{i_{k,2}}\sum_{\bar{i}=i_k\left(\tilde{\mathbf{r}}_{k}^{(m)}\right)+1}^{i}\tilde{H}_{k,i}\left(\tilde{\mathbf{r}}^{(m)}\right)\tilde{H}_{k,\bar{i}}\left(\tilde{\mathbf{r}}^{(m)}\right)\frac{(\alpha+\gamma)\delta}{s_k}.
\end{aligned}\tag{A.27}$$

In equation (A.27), the first equality results from the fact that the distribution of flow rates in the early arrival intervals is already at the UE state and hence $\tilde{H}_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}^{(m)}\right)=0$ holds for $\bar{k}=1,2,\cdots,K$ and $\bar{i}=1,2,\cdots,i_{\bar{k}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)$. The second equality is obtained by introducing formulae (11), (A.20), and (A.26) into its left hand side. The third equality is generated by introducing expression (A.21) into its left hand side.

By expression (9), the first term on the right hand side of equation (A.27) can be further deduced as

$$\begin{aligned}
&-\sum_{\bar{k}=1}^{K}\sum_{\bar{i}=i_{\bar{k}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)+1}^{M}\sum_{\bar{l}=1}^{K}\sum_{\bar{j}=i_{\bar{l}}\left(\tilde{\mathbf{r}}_{\bar{l}}^{(m)}\right)+1}^{M}\sum_{l=1}^{K}\sum_{j=i_l\left(\tilde{\mathbf{r}}_{l}^{(m)}\right)+1}^{M}G_{\bar{k}\bar{i},\bar{l}\bar{j}}\left(\tilde{\mathbf{r}}^{(m)}\right)\\
&\qquad\times\left(\left[C_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)-C_{l,j}\left(\tilde{\mathbf{r}}_{l}^{(m)}\right)\right]_{+}^{2}-\left[C_{\bar{l},\bar{j}}\left(\tilde{\mathbf{r}}_{\bar{l}}^{(m)}\right)-C_{l,j}\left(\tilde{\mathbf{r}}_{l}^{(m)}\right)\right]_{+}^{2}\right)\\
&\qquad\leq-\sum_{\bar{k}=1}^{K}\sum_{\bar{i}=i_{\bar{k}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)+1}^{M}\sum_{\bar{l}=1}^{K}\sum_{\bar{j}=i_{\bar{l}}\left(\tilde{\mathbf{r}}_{\bar{l}}^{(m)}\right)+1}^{M}\tilde{r}_{\bar{k},\bar{i}}^{(m)}\left[C_{\bar{k},\bar{i}}\left(\tilde{\mathbf{r}}_{\bar{k}}^{(m)}\right)-C_{\bar{l},\bar{j}}\left(\tilde{\mathbf{r}}_{\bar{l}}^{(m)}\right)\right]_{+}^{3},
\end{aligned}\tag{A.28}$$

and the second term can be further deduced as

$$\begin{aligned}
&-2\sum_{k=1}^{k_0}\sum_{i=i_k\left(\tilde{\mathbf{r}}_{k}^{(m)}\right)+1}^{i_{k,2}}\sum_{\bar{i}=i_k\left(\tilde{\mathbf{r}}_{k}^{(m)}\right)+1}^{i}\tilde{H}_{k,i}\left(\tilde{\mathbf{r}}^{(m)}\right)\tilde{H}_{k,\bar{i}}\left(\tilde{\mathbf{r}}^{(m)}\right)\frac{(\alpha+\gamma)\delta}{s_k}\\
&\qquad=-\sum_{k=1}^{k_0}\frac{(\alpha+\gamma)\delta}{s_k}\left(\left(\sum\nolimits_{i=i_k\left(\tilde{\mathbf{r}}_{k}^{(m)}\right)+1}^{i_{k,2}}\tilde{H}_{k,i}\left(\tilde{\mathbf{r}}^{(m)}\right)\right)^{2}+\sum\nolimits_{i=i_k\left(\tilde{\mathbf{r}}_{k}^{(m)}\right)+1}^{i_{k,2}}\tilde{H}_{k,i}\left(\tilde{\mathbf{r}}^{(m)}\right)^{2}\right).
\end{aligned}\tag{A.29}$$

If the distribution of flow rates in the late arrival intervals is at non-equilibrium state in iteration m, then, both the terms on the right hand side of equation (A.27) are negative. Thus, the inner product between the update direction of flow rates and the gradient of the function $V(\cdot)$ is negative, i.e., inequality (25) holds. This completes the proof.

A.6. Proof of Lemma 4

Proof. The derivative of V with respect to $\tilde{\lambda}$ is formulated as

$$dV(\tilde{\lambda})\Big/d\tilde{\lambda} = \nabla V\left(\tilde{\mathbf{r}}^{(m)} + \tilde{\lambda}\tilde{\mathbf{H}}\left(\tilde{\mathbf{r}}^{(m)}\right)\right)^{\mathrm{T}}\tilde{\mathbf{H}}\left(\tilde{\mathbf{r}}^{(m)}\right).$$

By Lemma 3, $dV(\tilde{\lambda})\Big/d\tilde{\lambda} < 0$ for $\tilde{\lambda} = 0$ in iteration m at non-equilibrium state. Additionally, $dV\Big/d\tilde{\lambda}$ is continuous with respect to $\tilde{\lambda} \in \left[0, \hat{\lambda}^{(m)}\right]$. Thus, there exists $\tilde{\lambda}_1^{(m)} \in \left(0, \hat{\lambda}^{(m)}\right]$ such that

$$dV(\tilde{\lambda})\Big/d\tilde{\lambda} < 0, \text{for } \tilde{\lambda} \in \left[0, \tilde{\lambda}_1^{(m)}\right), \text{and } dV(\tilde{\lambda})\Big/d\tilde{\lambda} = 0, \text{for } \tilde{\lambda} = \tilde{\lambda}_1^{(m)},$$

or

$$dV(\tilde{\lambda})\Big/d\tilde{\lambda} < 0, \text{for } \tilde{\lambda} \in \left[0, \hat{\lambda}^{(m)}\right], \text{and } \tilde{\lambda}_1^{(m)} = \hat{\lambda}^{(m)}.$$

As a result, the value of the function $V(\cdot)$ will decrease when set $\tilde{\lambda}^{(m)} = \tilde{\lambda}_1^{(m)}$ in each iteration m at non-equilibrium state until its value becomes zero. $V(\mathbf{r}) = 0$ holds if and only if $\mathbf{r} \in \Omega$ satisfies conditions (16) and (17). Thus, the flow rates in the late arrival intervals will converge to UE. This completes the proof.

A.7. Proof of Theorem 3

Proof. The necessity is first proved. If $\mathbf{r}^{(n+1)} = \bar{\mathbf{r}}^{(n)}$, then

$$\begin{aligned}
&\sum\nolimits_{k=1}^{K}\sum\nolimits_{i=1}^{M}\left(r_{k,i}^{(n+1)} - \bar{r}_{k,i}^{(n)}\right)C_{k,i}\left(\bar{\mathbf{r}}_k^{(n)}\right)\\
&\quad= \lambda^{(n)}\left(\sum\nolimits_{k=1}^{K}\sum\nolimits_{i=1}^{i_k\left(\bar{\mathbf{r}}_k^{(n)}\right)}\sum\nolimits_{l=1}^{K}\sum\nolimits_{j=i_l\left(\bar{\mathbf{r}}_l^{(n)}\right)+1}^{M}\left(G_{lj,ki}\left(\bar{\mathbf{r}}^{(n)}\right) - G_{ki,lj}\left(\bar{\mathbf{r}}^{(n)}\right)\right)C_{k,i}\left(\bar{\mathbf{r}}_k^{(n)}\right)\right.\\
&\qquad\left.+\sum\nolimits_{k=1}^{K}\sum\nolimits_{i=i_k\left(\bar{\mathbf{r}}_k^{(n)}\right)+1}^{M}\sum\nolimits_{l=1}^{K}\sum\nolimits_{j=1}^{i_l\left(\bar{\mathbf{r}}_l^{(n)}\right)}\left(G_{lj,ki}\left(\bar{\mathbf{r}}^{(n)}\right) - G_{ki,lj}\left(\bar{\mathbf{r}}^{(n)}\right)\right)C_{k,i}\left(\bar{\mathbf{r}}_k^{(n)}\right)\right)\\
&\quad= -\lambda^{(n)}\left(\sum\nolimits_{k=1}^{K}\sum\nolimits_{i=1}^{i_k\left(\bar{\mathbf{r}}_k^{(n)}\right)}\sum\nolimits_{l=1}^{K}\sum\nolimits_{j=i_l\left(\bar{\mathbf{r}}_l^{(n)}\right)+1}^{M}G_{ki,lj}\left(\bar{\mathbf{r}}^{(n)}\right)\left(C_{k,i}\left(\bar{\mathbf{r}}_k^{(n)}\right) - C_{l,j}\left(\bar{\mathbf{r}}_l^{(n)}\right)\right)\right.\\
&\qquad\left.+\sum\nolimits_{k=1}^{K}\sum\nolimits_{i=i_k\left(\bar{\mathbf{r}}_k^{(n)}\right)+1}^{M}\sum\nolimits_{l=1}^{K}\sum\nolimits_{j=1}^{i_l\left(\bar{\mathbf{r}}_l^{(n)}\right)}G_{ki,lj}\left(\bar{\mathbf{r}}^{(n)}\right)\left(C_{k,i}\left(\bar{\mathbf{r}}_k^{(n)}\right) - C_{l,j}\left(\bar{\mathbf{r}}_l^{(n)}\right)\right)\right) = 0. \qquad \text{(A.30)}
\end{aligned}$$

In equation (A.30), the first equality is obtained by substituting formulae (12) and (13) into its left hand side. Therefore, it follows that

$$\begin{aligned}
G_{ki,lj}\left(\bar{\mathbf{r}}^{(n)}\right) = \bar{r}_{k,i}^{(n)}\left[C_{k,i}\left(\bar{\mathbf{r}}_k^{(n)}\right) - C_{l,j}\left(\bar{\mathbf{r}}_l^{(n)}\right)\right]_+ = 0, \text{for } k = 1, 2, \cdots, K,\\
i = 1, 2, \cdots, i_k\left(\bar{\mathbf{r}}_k^{(n)}\right), l = 1, 2, \cdots, K, \text{and } j = i_l\left(\bar{\mathbf{r}}_l^{(n)}\right) + 1, \cdots, M, \qquad \text{(A.31)}
\end{aligned}$$

$$\begin{aligned}
G_{ki,lj}\left(\bar{\mathbf{r}}^{(n)}\right) = \bar{r}_{k,i}^{(n)}\left[C_{k,i}\left(\bar{\mathbf{r}}_k^{(n)}\right) - C_{l,j}\left(\bar{\mathbf{r}}_l^{(n)}\right)\right]_+ = 0, \text{for } k = 1, 2, \cdots, K,\\
i = i_k\left(\bar{\mathbf{r}}_k^{(n)}\right) + 1, \cdots, M, l = 1, 2, \cdots, K, \text{and } j = 1, 2, \cdots, i_l\left(\bar{\mathbf{r}}_l^{(n)}\right). \qquad \text{(A.32)}
\end{aligned}$$

In addition, by Theorem 2, $\bar{\mathbf{r}}^{(n)}$ satisfies expressions (16) and (17). Therefore, $\bar{\mathbf{r}}^{(n)}$ satisfies condition (26).

The sufficiency is then proved. If $\bar{\mathbf{r}}^{(n)}$ satisfies condition (26), then expressions (A.31) and (A.32) hold. By formulae (12) and (13), it is concluded that $\mathbf{r}^{(n+1)} = \bar{\mathbf{r}}^{(n)}$ holds. This completes the proof.

A.8. Proof of Lemma 5

Proof. Given the middle flow rate vector $\bar{\mathbf{r}}^{(n)}$, if there is queue at the bottleneck on link k, then the bottleneck on link k is fully utilized by early arrival commuters in the interval from $i_{k,1}^{(n)}\delta+T_k^f$ to t^* and by late arrival commuters in the interval from t^* to $i_{k,2}^{(n)}\delta+T_k^f$. Thus, there are relationships

$$t^*-i_{k,1}^{(n)}\delta-T_k^f\approx N_{k,1}^{(n)}/s_k \text{ and } i_{k,2}^{(n)}\delta+T_k^f-t^*\approx N_{k,2}^{(n)}/s_k\,. \tag{A.33}$$

The travel costs of all the commuters entering link k in the early (late) arrival interval are equal. Therefore, the minimum travel cost $\mu_1^{(n)}$ ($\mu_2^{(n)}$) of commuters during the early (late) arrival intervals equals the travel cost of the first (last) commuter entering link k, i.e.,

$$\mu_1^{(n)}\approx\beta\left(t^*-i_{k,1}^{(n)}\delta-T_k^f\right) \text{ and } \mu_2^{(n)}\approx\gamma\left(i_{k,2}^{(n)}\delta+T_k^f-t^*\right). \tag{A.34}$$

Substituting expression (A.33) into (A.34) immediately generates formula (27). This completes the proof.

A.9. Proof of Lemma 6

Proof. The case of $\mu_1^{(n)}<\mu_2^{(n)}$ is first examined. There is queue at the bottleneck on link k, and hence the sets $\left\{i_{k,1}^{(n)},\cdots,i_k^{(n)}\right\}$ and $\left\{i_k^{(n)}+1,\cdots,i_{k,2}^{(n)}\right\}$ are nonempty. For $i=1,\cdots,i_{k,1}^{(n)}-1$, $l=1,2,\cdots,K$, and $j=i_l^{(n)}+1,\cdots,M$, in virtue of $\bar{r}_{k,i}^{(n)}=0$, it is generated from formula (9) that

$$G_{ki,lj}\left(\bar{\mathbf{r}}^{(n)}\right)=\bar{r}_{k,i}^{(n)}\left[C_{k,i}\left(\bar{\mathbf{r}}_k^{(n)}\right)-C_{l,j}\left(\bar{\mathbf{r}}_l^{(n)}\right)\right]_+=0.$$

Thus, it is obtained from formula (13) that

$$H_{k,i}\left(\bar{\mathbf{r}}^{(n)}\right)=\sum\nolimits_{l=1}^{K}\sum\nolimits_{j=i_l^{(n)}+1}^{M}G_{lj,ki}\left(\bar{\mathbf{r}}^{(n)}\right)\geq 0, \text{for } i=1,\cdots,i_{k,1}^{(n)}-1. \tag{A.35}$$

For $i=i_{k,1}^{(n)},\cdots,i_k^{(n)}$, $l=1,2,\cdots,K$, and $j=i_l^{(n)}+1,\cdots,M$, due to

$$C_{k,i}\left(\bar{\mathbf{r}}_k^{(n)}\right)=\mu_1^{(n)}<\mu_2^{(n)}\leq C_{l,j}\left(\bar{\mathbf{r}}_l^{(n)}\right),$$

it is concluded from formula (9) that $G_{ki,lj}\left(\bar{\mathbf{r}}^{(n)}\right)=0$. For $i=i_{k,1}^{(n)},\cdots,i_k^{(n)}$, $l=1,2,\cdots,K$, and $j=i_l^{(n)}+1,\cdots,i_{l,2}^{(n)}$, in view of

$$\bar{r}_{l,j}^{(n)}>0 \text{ and } C_{k,i}\left(\bar{\mathbf{r}}_k^{(n)}\right)=\mu_1^{(n)}<\mu_2^{(n)}=C_{l,j}\left(\bar{\mathbf{r}}_l^{(n)}\right),$$

it is yielded from formula (9) that

$$G_{lj,ki}\left(\bar{\mathbf{r}}^{(n)}\right)=\bar{r}_{l,j}^{(n)}\left[C_{l,j}\left(\bar{\mathbf{r}}_l^{(n)}\right)-C_{k,i}\left(\bar{\mathbf{r}}_k^{(n)}\right)\right]_+>0.$$

Therefore, we have

$$H_{k,i}\left(\bar{\mathbf{r}}^{(n)}\right)\geq\sum\nolimits_{l=1}^{K}\sum\nolimits_{j=i_l^{(n)}+1}^{i_{l,2}^{(n)}}G_{lj,ki}\left(\bar{\mathbf{r}}^{(n)}\right)>0, \text{for } i=i_{k,1}^{(n)},\cdots,i_k^{(n)}. \tag{A.36}$$

For $i=i_k^{(n)}+1,\cdots,M$, $l=1,2,\cdots,K$, and $j=1,\cdots,i_{l,1}^{(n)}-1$, due to $\bar{r}_{l,j}^{(n)}=0$, $G_{lj,ki}\left(\bar{\mathbf{r}}^{(n)}\right)=0$. For $i=i_k^{(n)}+1,\cdots,M$, $l=1,2,\cdots,K$, and $j=i_{l,1}^{(n)},\cdots,i_l^{(n)}$, in virtue of $C_{l,j}\left(\bar{\mathbf{r}}_l^{(n)}\right)<C_{k,i}\left(\bar{\mathbf{r}}_k^{(n)}\right)$, $G_{lj,ki}\left(\bar{\mathbf{r}}^{(n)}\right)=0$. For $i=i_k^{(n)}+1,\cdots,i_{k,2}^{(n)}$, $l=1,2,\cdots,K$, and $j=i_{l,1}^{(n)},\cdots,i_l^{(n)}$, in view of $\bar{r}_{k,i}^{(n)}>0$ and $C_{k,i}\left(\bar{\mathbf{r}}_k^{(n)}\right)>C_{l,j}\left(\bar{\mathbf{r}}_l^{(n)}\right)$, $G_{ki,lj}\left(\bar{\mathbf{r}}^{(n)}\right)>0$. For $i=i_{k2}^{(n)}+1,\cdots,M$, $l=1,2,\cdots,K$, and $j=1,\cdots,i_l^{(n)}$, due to $\bar{r}_{k,i}^{(n)}=0$, $G_{ki,lj}\left(\bar{\mathbf{r}}^{(n)}\right)=0$. Thus, it follows that

$$H_{k,i}\left(\bar{\mathbf{r}}^{(n)}\right)\leq-\sum\nolimits_{l=1}^{K}\sum\nolimits_{j=i_{l,1}^{(n)}}^{i_l^{(n)}}G_{ki,lj}\left(\bar{\mathbf{r}}^{(n)}\right)<0, \text{for } i=i_k^{(n)}+1,\cdots,i_{k,2}^{(n)}, \tag{A.37}$$

$$H_{k,i}\left(\bar{\mathbf{r}}^{(n)}\right)=0, \text{for } i=i_{k,2}^{(n)}+1,\cdots,M. \tag{A.38}$$

Combining (A.35) to (A.38) generates formula (28). Similar to the above analyses, formula (29) can be proved. This completes the proof.

A.10. Proof of Theorem 4

Proof. Theorem 2 and Lemmas 1 to 4 indicate that the sub-iteration procedure of route flow rates in Step 2 of the two-stage iteration procedure converges to the middle flow rate vector $\bar{\mathbf{r}}^{(n)}$ and $\bar{\mathbf{r}}^{(n)}$ is at UE respectively in the early arrival intervals and the late arrival intervals. By Lemma 5, when the distribution of flows in the network is $\bar{\mathbf{r}}^{(n)}$, the minimum travel cost $\mu_1^{(n)}$ ($\mu_2^{(n)}$) of commuters in the early (late) arrival intervals is continuous and increasing with respect to the number $N_{k,1}^{(n)}$ ($N_{k,2}^{(n)}$) of commuters entering link k in the early (late) arrival interval if there is queue at the bottleneck on link k. It follows from equation (27) that

$$\mu_1^{(n)} \approx \frac{\beta \sum_{k=1}^{K} N_{k,1}^{(n)}}{\sum_{k=1}^{K} s_k} \text{ and } \mu_2^{(n)} \approx \frac{\gamma \sum_{k=1}^{K} N_{k,2}^{(n)}}{\sum_{k=1}^{K} s_k}. \tag{A.39}$$

Without loss of generality, it is assumed that $\mu_1^{(n)} < \mu_2^{(n)}$ holds. It is then known from Lemma 6 that the number of commuters entering link k in the early arrival interval will increase and the number of commuters entering link k in the late arrival interval will decrease in the next iteration. In addition, as formulated by expressions (10) and (11), the update of flow rates in Step 2 satisfies the flow conservation constraint respectively in the early arrival intervals and the late arrival intervals. Thus, the minimum travel cost $\mu_1^{(n+1)}$ in the early arrival intervals will ascend and the minimum travel cost $\mu_2^{(n+1)}$ in the late arrival intervals will descend in the next iteration. As a result, the differences $\mu_2^{(n)} - \mu_1^{(n)}$ and $\mu_2^{(n+1)} - \mu_1^{(n+1)}$ satisfy

$$\mu_2^{(n)} - \mu_1^{(n)} > \mu_2^{(n+1)} - \mu_1^{(n+1)}. \tag{A.40}$$

On the other hand, a small portion of flow rates in the early arrival intervals may transfer to the late arrival intervals due to queue delay, and hence we have

$$\sum_{k=1}^{K} N_{k,1}^{(n+1)} \le \delta \sum_{k=1}^{K} \sum_{i=1}^{i_k^{(n)}} \bar{r}_{k,i}^{(n+1)} = \sum_{k=1}^{K} N_{k,1}^{(n)} + \lambda^{(n)} \delta \sum_{k=1}^{K} \sum_{i=1}^{i_k^{(n)}} H_{k,i}\left(\bar{\mathbf{r}}^{(n)}\right), \tag{A.41}$$

$$\sum_{k=1}^{K} N_{k,2}^{(n+1)} \ge \delta \sum_{k=1}^{K} \sum_{i=i_k^{(n)}+1}^{M} \bar{r}_{k,i}^{(n+1)} = \sum_{k=1}^{K} N_{k,2}^{(n)} + \lambda^{(n)} \delta \sum_{k=1}^{K} \sum_{i=i_k^{(n)}+1}^{M} H_{k,i}\left(\bar{\mathbf{r}}^{(n)}\right). \tag{A.42}$$

In equations (A.41) and (A.42), the equalities are generated by substituting formula (12) into their left hand side. By expression (A.39), it is generated that

$$\begin{aligned}
\mu_2^{(n+1)} - \mu_1^{(n+1)} &\approx \frac{\gamma \sum_{k=1}^{K} N_{k,2}^{(n+1)}}{\sum_{k=1}^{K} s_k} - \frac{\beta \sum_{k=1}^{K} N_{k,1}^{(n+1)}}{\sum_{k=1}^{K} s_k} \\
&\ge \frac{\gamma \sum_{k=1}^{K} N_{k,2}^{(n)}}{\sum_{k=1}^{K} s_k} + \frac{\lambda^{(n)} \delta \gamma \sum_{k=1}^{K} \sum_{i=i_k^{(n)}+1}^{M} H_{k,i}\left(\bar{\mathbf{r}}^{(n)}\right)}{\sum_{k=1}^{K} s_k} - \frac{\beta \sum_{k=1}^{K} N_{k,1}^{(n)}}{\sum_{k=1}^{K} s_k} - \frac{\lambda^{(n)} \delta \beta \sum_{k=1}^{K} \sum_{i=1}^{i_k^{(n)}} H_{k,i}\left(\bar{\mathbf{r}}^{(n)}\right)}{\sum_{k=1}^{K} s_k} \\
&= \mu_2^{(n)} - \mu_1^{(n)} + \frac{\lambda^{(n)} \delta \left(\gamma \sum_{k=1}^{K} \sum_{i=i_k^{(n)}+1}^{M} H_{k,i}\left(\bar{\mathbf{r}}^{(n)}\right) - \beta \sum_{k=1}^{K} \sum_{i=1}^{i_k^{(n)}} H_{k,i}\left(\bar{\mathbf{r}}^{(n)}\right) \right)}{\sum_{k=1}^{K} s_k} \ge 0.
\end{aligned} \tag{A.43}$$

In equation (A.43), the first inequality is obtained by substituting inequalities (A.41) and (A.42) into its left hand side. The second inequality is generated by relationships (28) and (30). Inequalities (A.40) and (A.43) indicate that the difference between the minimum travel costs respectively in the early arrival and late arrival intervals becomes decreasing as the number of iterations increases. As a result, the flow rate vector sequence $\left\{\bar{\mathbf{r}}^{(n)}, n = 0, 1, 2, \cdots\right\}$ generated by the two-stage iteration procedure converges to the UE solution. This completes the proof.

Appendix B: Symbols Used in This Paper

B.1. Symbols Involved in Section 2

N: The number of commuters traveling between the OD pair

K: The number of parallel links between the OD pair

s_k: The capacity or service rate of the bottleneck on link k

$[0, \bar{T}]$: A sufficiently long morning rush hour

M: The number of time slices which the time horizon $[0, \bar{T}]$ is discretized into

δ: The length of a time slice

$\mathbf{r}$: The flow rate vector

$r_{k,i}$: The flow rate entering link k at time slice i

Ω: The set of feasible flow rate vectors

$\mathbf{r}_k$: The flow rate vector of link k

$T_{k,i}(\mathbf{r}_k)$: The travel time of commuters entering link k at time slice i

T_k^f: The free flow travel time of link k

$T_{k,i}^v(\mathbf{r}_k)$: The queuing time of commuters entering link k at time slice i

$D_{k,i}(\mathbf{r}_k)$: The length of the queue at the bottleneck on link k when commuters entering link k at time slice i arrive at the bottleneck

t^*: The preferred arrival time of all commuters

$C_{k,i}(\mathbf{r}_k)$: The travel cost of commuters entering link k at time slice i

α: The value of travel time

β: The value of schedule delay early

γ: The value of schedule delay late

$\mathbf{C}(\mathbf{r})$: The travel cost vector

μ: The minimum travel cost of commuters

$G_{ki,lj}(\mathbf{r})$: The flow rate transfer from link and time slice (k, i) to link and time slice (l, j)

$\tilde{\lambda}^{(m)}$: An update step size of flow rates in Step 2.2 of the two-stage computing procedure

$\hat{\lambda}^{(m)}$: An upper bound of the update step size $\tilde{\lambda}^{(m)}$

$\tilde{\varepsilon}$: A smaller positive number of convergence tolerance in Step 2.3 of the two-stage computing procedure

$\lambda^{(n)}$: An update step size of flow rates in Step 3 of the two-stage computing procedure

$\bar{\lambda}^{(n)}$: An upper bound of the update step size $\lambda^{(n)}$

ε: A smaller positive number of convergence tolerance in Step 4 of the two-stage computing procedure

μ_1: The minimum travel cost of commuters during the early arrival intervals

μ_2: The minimum travel cost of commuters during the late arrival intervals

$V(\cdot)$: A gap function with respect to $\mathbf{r}$

$\eta^{(n)}$: An update step size of flow rates in iteration n of the proportional swap system

$\bar{V}(\cdot)$: A gap function with respect to $\mathbf{r}$

κ: A larger positive number (at least, larger than one) in Step 1 of the GS method

ξ: A smaller positive number of convergence tolerance in Step 3 of the GS method

$\underline{\lambda}$: A smaller positive number for determining the update step size $\lambda^{(n)}$

λ_0: A smaller positive number for determining the update step size $\tilde{\lambda}^{(m)}$

B.2. Symbols Involved in Section 3

U: The number of user classes

N_u: The number of commuters in class u

s: The capacity or service rate of the bottleneck

$r_{u,i}$: The flow rate of user class u entering the link at time slice i

$T_i(\mathbf{r})$: The travel time of commuters entering the link at time slice i

T^f: The free flow travel time of the link

$T_i^v(\mathbf{r})$: The queuing time of commuters entering the link at time slice i

$D_i(\mathbf{r})$: The length of the queue at the bottleneck when commuters entering the link at time slice i arrive at the bottleneck

t_u^*: The preferred arrival time of user class u

$C_{u,i}(\mathbf{r})$: The travel cost of user class u entering the link at time slice i

α_u: The value of travel time of user class u

β_u: The value of schedule delay early of user class u

γ_u: The value of schedule delay late of user class u

μ_u: The minimum travel cost of user class u

$G_{u,i,j}(\mathbf{r})$: The flow rate transfer of user class u from time slice i to j

$\mu_{1,u}$: The minimum travel cost of user class u during the early arrival interval

$\mu_{2,u}$: The minimum travel cost of user class u during the late arrival interval