Electronic Companion

EC.1. Preliminaries

As the application of information design in transportation is relatively new, we cover in this section its basics, through an illustrative scenario with two players: a Sender (he) and a Receiver (she). The game is set in a world characterized by a state $w \in \mathbb{W}$, where $\mathbb{W}$ is a finite set. We assume $\boldsymbol{\mu}_0 = \{\mu_0(w)\}_{w\in\mathbb{W}} \in \boldsymbol{\Delta}(\mathbb{W})$ is a vector that represents the Receiver's prior belief about each possible world state. Note that $\boldsymbol{\Delta}$ denotes a probability simplex, i.e., the sum of vector elements is 1. The Receiver has a utility function $u(a, w)$ that depends on her action $a \in \mathbb{A}$ and the world state $w \in \mathbb{W}$. The Sender has his own utility function $v(a, w)$ that also depends on the Receiver's a and w. We assume both $u(a, w)$ and $v(a, w)$ are continuous in a.

The Receiver's action depends on her belief about the world, and the Sender can try to manipulate that belief to maximize his own utility. To do so, the Sender may choose among a finite set of signals $\mathbb{S}$, according to an information structure $\pi : \mathbb{S} \times \mathbb{W} \to [0, 1]$, where $\pi(s \mid w)$ gives the probability that a world state w corresponds to a signal state s. Clearly, we have $\sum_{s\in\mathbb{S}} \pi(s \mid w) = 1$ $(\forall w \in \mathbb{W})$. The Sender shares $\boldsymbol{\pi} = \{\pi(s \mid w)\}_{s\in\mathbb{S}, w\in\mathbb{W}}$, with the Receiver, and commits to it before the world state w is realized. Then, for each realized world state, the Sender sends to the Receiver a signal state s rendered based on the information structure $\boldsymbol{\pi}$.

EC.1.1. General information structure

Under a general information structure, upon receiving s, the Receiver forms a posterior belief $\boldsymbol{\mu}_s = \{\mu_s(w)\}_{w\in\mathbb{W}}$ using the Bayes' rule and chooses an action to maximize her expected utility. Specifically, the probability of the world state w, as perceived by the Receiver upon receiving a signal state s, is $\mu_s(w) = \frac{\mu_0(w)\pi(s|w)}{\sum_{w'\in\mathbb{W}} \mu_0(w')\pi(s|w')}$ $(\forall w \in \mathbb{W}, s \in \mathbb{S})$. The optimal action(s) of the Receiver can be written as a set $a^*(\boldsymbol{\mu}_s) = \arg\max_{a\in\mathbb{A}} \mathbb{E}_{w\sim\boldsymbol{\mu}_s}(u(a, w)) = \arg\max_{a\in\mathbb{A}} \sum_{w\in\mathbb{W}} \mu_s(w)u(a, w)$. For notational convenience, let $\hat{a}_s$ be the best response action from $a^*(\boldsymbol{\mu}_s)$ that is preferred by the Sender. Accordingly, the Sender's objective is the maximization of his expected utility

$$V(\boldsymbol{\pi}) = \mathbb{E}_{w\sim\boldsymbol{\mu}_0}\mathbb{E}_{s\sim\pi(\cdot|w)} v(\hat{a}_s, w) = \sum_{w\in\mathbb{W}} \mu_0(w) \sum_{s\in\mathbb{S}} \pi(s \mid w)v(\hat{a}_s, w). \tag{EC.19}$$

Equation (EC.19) indicates that the information structure $\boldsymbol{\pi}$ affects the Sender's payoff directly as well as indirectly, by influencing the Receiver's best response action $\hat{a}_s \in a^*(\boldsymbol{\mu}_s)$ (which depends on the posterior belief $\boldsymbol{\mu}_s$). Accordingly, the Sender's optimization problem can be formulated as follows, with $\boldsymbol{\pi}$ as the decision variable:

$$\max\ V(\boldsymbol{\pi}) \quad \text{subject to:} \sum_{s\in\mathbb{S}} \pi(s \mid w) = 1, \forall w \in \mathbb{W}; \pi(s \mid w) \geq 0, \forall s \in \mathbb{S}, w \in \mathbb{W}. \tag{EC.20}$$

EC.1.2. Direct information structure

Before solving Problem (EC.20), the Sender must choose an appropriate signal set $\mathbb{S}$. For this task, according to the revelation principle (Myerson 1979, 1986, Bergemann and Morris 2019, Gan et al. 2024), the Sender may focus on direct signaling schemes. A direct information structure assumes $\mathbb{S} = \mathbb{A}$ so that each signal now corresponds to an action recommendation. More specifically, $\pi(a \mid w)$ specifies the probability of sending an action recommendation a according to the realized state w. The Receiver will always follow the recommendation a if it is optimal under the posterior belief, i.e., $\sum_{w\in\mathbb{W}}\mu_0(w)\,\pi(a\mid w)\,[u(a,w)-u(a',w)]\geq 0, \forall a'\in\mathbb{A}$, which are known as the persuasive or obedience constraints. Thus, the Sender's optimization problem can be formulated as follows (Dughmi and Xu 2016):

$$\max_{\boldsymbol{\pi}} \sum_{w\in\mathbb{W}}\sum_{a\in\mathbb{A}}\mu_0(w)\pi(a\mid w)v(a,w) \tag{EC.21a}$$

$$\text{subject to: } \sum_{w\in\mathbb{W}}\mu_0(w)\,\pi(a\mid w)\,[u(a,w)-u(a',w)]\geq 0, \forall a,a'\in\mathbb{A}, \tag{EC.21b}$$

$$\sum_{a\in\mathbb{A}}\pi(a\mid w)=1, \forall w\in\mathbb{W};\ \pi(a\mid w)\geq 0, \forall a\in\mathbb{A}, w\in\mathbb{W}. \tag{EC.21c}$$

Here, the objective is the Sender's expected utility, Constraints (EC.21b) enforce the Receiver's obedience to recommendation and Constraints (EC.21c) ensure the feasibility of the information structure.

EC.2. Table of notations

Table 4: Notations.

Notation	Definition
	— Base models general settings. —
$a\in\mathbb{A}$	An action a from the action set $\mathbb{A}$.
$\mathbb{A}=\{C,M\}$	Action set $\mathbb{A}$ consisting of Route C (charging) and Route M (non-charging).
$w\in\mathbb{W}$	A state w from the state set $\mathbb{W}$.
$\boldsymbol{\mu}_0=\{\mu_0(w)\}_{w\in\mathbb{W}}$	Prior belief over all possible state $w\in\mathbb{W}$.
l	The length of the remaining journey from the decision point.
Δ_l	The detour length for using the charging station.
v	The travel speed of the drivers.
$\Delta_t\equiv\Delta_l/v$	Detour expressed in travel time units.
r	The remaining range estimate (RRE) at the decision point.
$p(r)$	The probability of completing the journey without charging.
$g(\cdot)$	The probability density function (PDF) of the RRE distribution across the population.
$G(\cdot)$	The cumulative distribution function (CDF) of the RRE distribution across the population.
$\Xi=[\underline{R},\bar{R}]$	The support of the RRE distribution.
f	Flow of drivers choosing Route C, also referred to as the charging flow.
$1-f$	Flow of drivers choosing Route M.
f^*	Bayesian Nash equilibrium (BNE) flow choosing Route C.
$J_C(f\mid w)$	The cost function associated with Route C at charging flow level f at state w.
$J_M(p(r)\mid w)$	The cost function associated with Route M for driver with RRE r at state w.
$\bar{r}$	Threshold RRE that makes the driver indifferent between C and M under charging flow f.
$\bar{r}^*$	BNE threshold RRE indifferent between choosing Route C and M.
	— Public information design base model. —
$s\in\mathbb{S}$	A signal s from the signal set $\mathbb{S}$.
$\boldsymbol{\pi}=\{\pi(s\mid w)\}_{s\in\mathbb{S},w\in\mathbb{W}}$	Information structure specifying the probability of sending signal s given state w.
$\boldsymbol{\mu}_s=\{\mu_s(w)\}_{w\in\mathbb{W}}$	Posterior belief over states after receiving signal s.
f_s^*	BNE flow choosing Route C after receiving public signal s.
$\bar{r}_s^*$	BNE threshold RRE under public signal s, indifferent between C and M under posterior $\boldsymbol{\mu}_s$.
$Z(\boldsymbol{\pi})$	Expected total system cost for the public information design given $\boldsymbol{\pi}$.
	— Private information design base model. —
$\boldsymbol{\pi}=\{\pi(a\mid w,r)\}_{a\in\mathbb{A},w\in\mathbb{W},r\in\Xi}$	An information structure specifying the probability to recommend action a for r at w.
$\bar{\boldsymbol{r}}=\{\bar{r}_w\}_{w\in\mathbb{W}}$	A vector of state-dependent cutoff RRE thresholds for private information design.
$\{\bar{r}_i\}_{i=1,\ldots,\lvert\mathbb{W}\rvert}$	A vector of sorted state-dependent cutoff RRE thresholds for private information design.
$\boldsymbol{f}=\{f_w\}_{w\in\mathbb{W}}$	A vector of state-dependent charging flows for private information design.
$\boldsymbol{\nu}=\{\nu_i^w\}_{i=1,\ldots,\lvert\mathbb{W}\rvert,w\in\mathbb{W}}$	Indicator $\nu_i^w=1$ if $\bar{r}_w$ is the i-th smallest cutoff; 0 otherwise.
$Z(\boldsymbol{f})$ or $Z(\boldsymbol{f},\boldsymbol{\nu})$	Expected total system cost for the private information design given $\boldsymbol{f}$ (and indicators $\boldsymbol{\nu}$).
$\hat{\boldsymbol{r}}=\{\underline{R},\bar{r}_1,\ldots,\bar{r}_i,\ldots,\bar{r}_{\lvert\mathbb{W}\rvert},\bar{R}\}$	The sorted RRE thresholds augmented with the lower and upper bounds on RRE.

$\equiv \{\hat{r}_0, \hat{r}_1, \ldots, \hat{r}_k, \ldots, \hat{r}_{\lvert\mathbb{W}\rvert}, \hat{r}_{\lvert\mathbb{W}\rvert+1}\}$	
$R_k = (\ \hat{r}_{k-1}, \hat{r}_k\ ]$	RRE interval of type k travelers.
— Parameters used in the base models for analytical results and numerical experiments. —	
$\mathbb{W} = \{\mathrm{H}, \mathrm{L}\}$	State H denotes high grid load / adverse conditions; State L denotes normal operation.
α	Coefficient capturing the congestion effect.
β	Additional delay on Route C when $w = \mathrm{H}$.
ζ	Penalty coefficient for Route M.
n	Safe range factor defining the RRE distribution.
Superscript NI	Refers to no information benchmark.
Superscript FI	Refers to full information benchmark.
Superscript SO	Refers to system optimal benchmark.
Superscript PVI	Refers to the private information design.
$\underline{\mu_0}$	Constant given by $(\alpha\,(\zeta - \Delta_t))/(-\alpha\,(\zeta - \Delta_t) + \beta\,(2\alpha + \zeta))$.
$\overline{\mu_0}$	Constant given by $((\alpha + \zeta)\,(\zeta - \Delta_t))/(-\alpha\,(\zeta - \Delta_t) + \beta\,(2\alpha + \zeta))$.
$\hat{\mu_0}$	Constant given by $((\alpha^2 + (\alpha + \zeta)^2)(\zeta - \Delta_t))/(-\alpha\zeta\,(\zeta - \Delta_t) + \beta(2\alpha + \zeta)(\alpha + \zeta))$.
α_w	Congestion coefficient associated with state w (used in numerical experiments).
β_w	Additional delay associated with state w (used in numerical experiments).
— Two-station corridor extension general settings: $i = 1, 2$. —	
D_i	Decision point D_i on the two-station corridor.
$a_i \in \mathbb{A}_i$	Action a_i taken at decision point D_i, where $a_i \in \mathbb{A}_i$.
$\boldsymbol{a} = a_1 a_2 \in \mathbb{A}$	Driver's overall action sequence $\boldsymbol{a} = a_1 a_2$, where $\boldsymbol{a} \in \mathbb{A}$.
$\mathbb{A} = \{CC, CM, MC, MM\}$	Set of all possible action sequences.
$iw \in \mathbb{W}_i$	A state iw in state set $\mathbb{W}_i$ for Station i.
μ_{iw}^0	Prior belief of each state iw for Station i.
l_1	The distance between the two charging stations.
l_2	The distance from Station 2 to the final destination.
$g(\cdot)$	PDF of the initial RRE distribution.
$G(\cdot)$	CDF of the initial RRE distribution.
$u(\cdot)$	PDF of random range reduction per mile traveled, with support $\left[\underline{\delta}, \bar{\delta}\right]$.
$p(\hat{r}, l) = 1 - \int_{\hat{r}/l}^{\bar{\delta}} u(\delta)\,\mathrm{d}\delta$	Probability that a vehicle with RRE $\hat{r}$ can successfully cover distance l.
$J_i^p\,(\hat{r})$	Expected penalty for attempting to traverse l_i $(i = 1, 2)$ with RRE $\hat{r}$.
$\hat{r}_1^{a_1}(r), a_1 \in \{C, M\}$	Resulting RRE at D_1 after choosing the first action a_1 with the initial RRE r.
$\hat{r}_2^{\boldsymbol{a}}(r), \boldsymbol{a} \in \mathbb{A}$	Resulting RRE at D_2 after choosing the action sequence $\boldsymbol{a} \in \mathbb{A}$ with the initial RRE r.
— Two-station corridor public information design with sequential release. —	
$is \in \mathbb{S}_i$	A signal is at decision point D_i from the set of signals $\mathbb{S}_i$.
$\boldsymbol{\pi} = \{\boldsymbol{\pi}_1, \boldsymbol{\pi}_2\}$	Information structure vector for two stations.
$\boldsymbol{\pi}_i = \{\pi_i\,(is \mid iw)\}_{is \in \mathbb{S}_i, iw \in \mathbb{W}_i}$	Public information structure at decision point D_i.
$R_{1s}^{a_1 a_2}$	RRE range of drivers who choose actions $a_1 a_2$ after receiving signal $1s$.
R_{1s}^{C}	RRE range of drivers who choose to charge at Station 1 after receiving signal $1s$.
R_{1s}^{M}	RRE range of drivers who choose to skip at Station 1 after receiving signal $1s$.
$R_{1s2s}^{\boldsymbol{a}}$	RRE range of drivers who choose action sequence $\boldsymbol{a}$ after receiving signals $1s$ and $2s$.
$\mathbf{R}$	$\mathbf{R} = \left\{R_{1s}^{C}, R_{1s}^{M}, R_{1s2s}^{CC}, R_{1s2s}^{CM}, R_{1s2s}^{MC}, R_{1s2s}^{MM}\right\}_{1s \in \mathbb{S}_1, 2s \in \mathbb{S}_2}$ A set of ranges.
$J_{1C,1w}^{g,1s}$	Charging cost at Station 1 in state $1w \in \mathbb{W}_1$ after receiving signal $1s \in \mathbb{S}_1$.
$J_{a_1C,2w}^{g,1s2s}$	Charging cost at Station 2, conditional on $(1s, a_1)$, given signal $2s$ and state $2w$.
α_{iw}	The congestion coefficient related to state iw at Station i.
β_{iw}	The extra delay related to state iw at Station i.
μ_{1w}^{1s}	Posterior belief about state $1w$ for Station 1 after receiving signal $1s$.
μ_{2w}^{2s}	Posterior belief about state $2w$ for Station 2 after receiving signal $2s$.
$e_{a_1 a_2}^{1s}$	The expected costs for action sequence $a_1 a_2$ at D_1 after receiving signal $1s$.
$e_{a_1 a_2}^{1s2s}$	The expected costs for action a_2 at D_2, conditional on $(1s, a_1)$, after receiving signal $2s$.
$\varepsilon(\boldsymbol{\pi})$	BNE set for public information structure $\boldsymbol{\pi}$ for sequential release.
$Z_b(\mathbf{R}^*, \boldsymbol{\pi})$	The system's total expected cost (in sequential release public design) achieved at $\boldsymbol{\pi}$ and $\mathbf{R}^*$.
p_{1s}	The probability of observing each first-stage signal $1s$: $p_{1s} \equiv \sum_{1w \in \mathbb{W}_1} \mu_{1w}^0 \cdot \pi_1\,(1s \mid 1w)$.
— Two-station corridor private information design with sequential release. —	

$t \in \mathbb{T}$	A private design information type t belongs to $\mathbb{T}$ where $\lvert\mathbb{T}\rvert = 2^{(\sum_i \lvert\mathbb{W}_i\rvert)}$.
$\eta_{t,w}$	Type-state recommendation indicator, = 1 if type t receives C at state w, and 0 otherwise.
$\mathbb{K} = \{1, \ldots, K\}$	A set of intervals the range support Ξ is divided into.
$k \in \mathbb{K}$	The kth interval.
$\bar{r}_k$	The upper bound of the kth interval.
$\bar{\boldsymbol{r}} = [\ldots, \bar{r}_k, \ldots]$	The upper bound vector of all intervals.
$\iota_{k,t}$	Binary variable indicating whether drivers in the kth range interval are information type t.
$\boldsymbol{\iota} = [\ldots, \iota_{k,t}, \ldots]$	All interval-type indicating binary variables.
f_{1w}^{1C}	The charging flow at Station 1 under state $1w$.
f_{1w2w}^{CC}	The flow of charging at both stations given state $(1w, 2w)$.
f_{1w2w}^{MC}	The flow of only charging at Station 2 given state $(1w, 2w)$.
$J_{1C,1w}^{g}$	The charging cost at Station 1 under state $1w$ which depends on f_{1w}^{1C}.
$J_{CC,1w2w}^{g}$	Charging cost at Station 2 under action CC and states $(1w, 2w)$, dependent on f_{1w2w}^{CC}.
$J_{MC,1w2w}^{g}$	Charging cost at Station 2 under action MC and states $(1w, 2w)$, dependent on f_{1w2w}^{MC}.
$Z_v(\bar{\boldsymbol{r}}, \boldsymbol{\iota})$	The system's total expected cost (in sequential release private design) achieved at $\bar{\boldsymbol{r}}$ and $\boldsymbol{\iota}$.
$E_{1w}^{a_1}(r)$	The expected cost for RRE r at D_1 given state $1w$ and action a_1.
$E_{1w2w}^{\boldsymbol{a}}(r)$	The expected cost for RRE r at D_2 given states $(1w, 2w)$ and action sequence $\boldsymbol{a}$.
— Two-station corridor public information design with simultaneous release. —	
$\hat{\mathbf{R}}$	$\hat{\mathbf{R}} = \left\{\hat{R}_{1s2s}^{CC}, \hat{R}_{1s2s}^{CM}, \hat{R}_{1s2s}^{MC}, \hat{R}_{1s2s}^{MM}\right\}_{1s\in\mathbb{S}_1, 2s\in\mathbb{S}_2}$ A set of ranges for simultaneous release.
$\hat{\varepsilon}(\boldsymbol{\pi})$	BNE set for public information structure $\boldsymbol{\pi}$ for simultaneous release.
$\rho_{\boldsymbol{a}}^{1s2s}$	The expected cost for simultaneous release given public signals $(1s, 2s)$ and action sequence $\boldsymbol{a}$.
$\hat{J}_{1C,1w}^{g,1s2s}$	The charging cost at Station 1 given state $1w$ and public signals $(1s, 2s)$.
$\hat{J}_{a_1C,2w}^{g,1s2s}$	The charging cost at Station 2 given state $2w$ and public signals $(1s, 2s)$ and action a_1.
$\hat{Z}_b\left(\hat{\mathbf{R}}^*, \boldsymbol{\pi}\right)$	The system's total expected cost (in simultaneous release public design) achieved at $\boldsymbol{\pi}$ and $\mathbf{R}^*$.
— Two-station corridor private information design with simultaneous release. —	
$V_{\boldsymbol{a},\boldsymbol{a}'}^{k,t}(r)$	Expected cost for a type-t driver with RRE r in interval k, receiving recommendation $\boldsymbol{a}$ and choosing to take action $\boldsymbol{a}'$.
$\hat{Z}_v(\bar{\boldsymbol{r}}, \boldsymbol{\iota})$	The system's total expected cost (in simultaneous release private design) achieved at $\bar{\boldsymbol{r}}$ and $\boldsymbol{\iota}$.
f_{1w2w}^{1C}	The flow of charging at Station 1 under states $(1w, 2w)$.
$J_{1C,1w2w}^{g}$	Charging cost at Station 1 under states $(1w, 2w)$, dependent on f_{1w2w}^{1C}.
— Parameters for two-station corridor extension. —	
Δ	The fixed mileage gains from charging at each station.
$\delta_1 = l_1 \cdot \int_{\underline{\delta}}^{\bar{\delta}} \delta \cdot u(\delta)\, \mathrm{d}\delta$	The expected range reduction from traveling between the two stations.

EC.3. Proof of Theorem 1

To establish the equivalence between Problems (6) and (5), we first demonstrate the equivalence of their objective functions. Plugging the proposed information structure (4) into (5a), we have the following which equals (6a).

$$\sum_{w\in\mathbb{W}} \mu_0(w) \int_{r\in\Xi} \left[\pi(C|w,r) J_C(f_w|w) + \pi(M|w,r) J_M(p(r)|w)\right] g(r)\,\mathrm{d}r = \sum_{w\in\mathbb{W}} \mu_0(w) \left(\int_{\underline{R}}^{G^{-1}(f_w)} J_C(f_w|w)\, g(r)\,\mathrm{d}r + \int_{G^{-1}(f_w)}^{\bar{R}} J_M(p(r)|w)\, g(r)\,\mathrm{d}r\right) = \sum_{w\in\mathbb{W}} \mu_0(w) \left(f_w \cdot J_C(f_w|w) + \int_{G^{-1}(f_w)}^{\bar{R}} J_M(p(r)|w)\, g(r)\,\mathrm{d}r\right).$$

Recall that the ordered cutoff threshold values are denoted as $\bar{r}_i, i \in 1, \ldots, |\mathbb{W}|$ and a binary variable v_i^w (see (6i)) is used to indicate whether the rank of the cutoff threshold for state w is i (i.e., the i-th smallest among all cutoff thresholds). In Problem (5), Constraint (5d) establishes a one-to-one correspondence between the charging flow f_w and an RRE threshold $\bar{r}_w$ at state w. This relationship is defined in Constraints (6e) using the order indicator variables

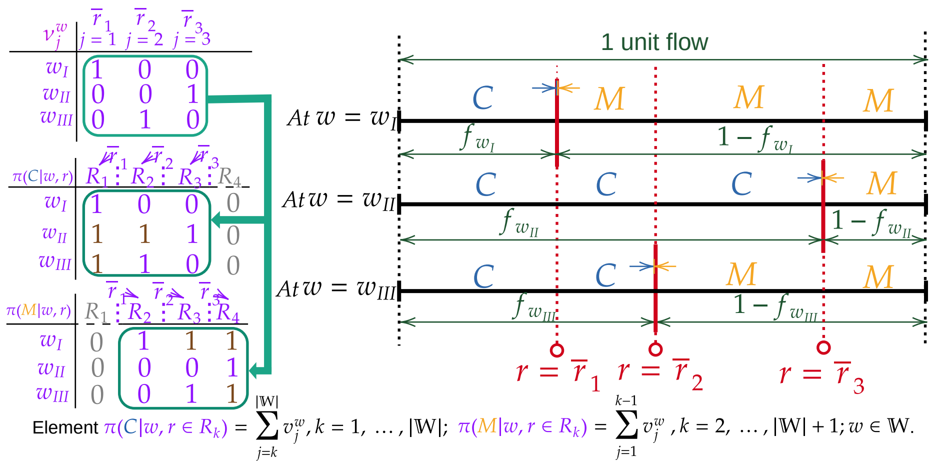


Figure 13 **Illustration of the order thresholds and the corresponding information structure.**

as $\bar{r}_i = G^{-1}\left(\sum_{w\in\mathbb{W}} \nu_i^w f_w\right)$, $i = 1, \ldots, |\mathbb{W}|$, with the order being enforced in Constraints (6d), i.e., $\sum_{w\in\mathbb{W}} \nu_i^w f_w \le \sum_{w\in\mathbb{W}} \nu_{i+1}^w f_w$, $i = 1, \ldots, |\mathbb{W}| - 1$.

We next demonstrate the equivalence between the obedience constraints in Problems (6), i.e., Constraints (6f)–(6g), and those in Problem (5), i.e., Constraints (5b)–(5c). The reader is referred to Figure 13 for illustration. First of all, recall that the RRE of the type k drivers, who receive the same recommendation at each state, falls into R_k, $k = 1, \ldots, |\mathbb{W}| + 1$, per the information structure imposed by (4). For any type k driver, only the cost associated with Route M, $J_M\left(p\left(r\right)|w\right)$ (see (5b)-(5c)), depends on r. Since $p\left(r\right)$ is an increasing function of r, and the cost of Route M is a decreasing function of $p\left(r\right)$, it suffices to check the compliance with a charging recommendation (Constraint (5b)) for the driver with the highest RRE in R_k (i.e., the upper bound, or $r = \hat{r}_k$), and the compliance with a no-charge recommendation (Constraint (5c)) for the driver with the lowest RRE in R_k (i.e., the lower bound or $r = \hat{r}_{k-1}$). For types 1 and $|\mathbb{W}| + 1$, the recommendation is charge and skip, respectively, for all states. Therefore, the obedience to charge can be ensured for everyone in type $k = 1, \ldots, |\mathbb{W}|$ by enforcing the obedience on the driver with $r = \hat{r}_k$, while the obedience to skip can be ensured for all type $k = 2, \ldots, |\mathbb{W}| + 1$ drivers by enforcing it on the driver with $r = \hat{r}_{k-1}$.

We proceed to show how the orders of the cutoff thresholds are enforced in the obedience constraints (6f)-(6g). The information structure (4) dictates that, at a given state w, everyone with an $r \le \bar{r}_w$ is recommended to charge, while others are recommended to skip. This means that $\pi(C \mid w, r \in R_k) = 1$ for any type $k \in [1, l]$ where $\nu_l^w = 1$, and $\pi(M \mid w, r \in R_k) = 1$ for any type $k \in (l, |\mathbb{W}| + 1]$ where $\nu_l^w = 1$, see Figure 13. Since $\boldsymbol{\nu}$ is endogenous, we set $\pi(C \mid w, r \in R_k) = \sum_{j=k}^{|\mathbb{W}|} \nu_j^w, \forall k = 1, \ldots, |\mathbb{W}|$, and $\pi(M \mid w, r \in R_k) = \sum_{j=1}^{k-1} \nu_j^w, \forall k = 2, \ldots, |\mathbb{W}| + 1$.

As a result, by capturing obedience at the cutoff points and replacing the information structure with the elements derived in the "state-interval" matrices, Constraints (5b)–(5c) are transformed into Constraints (6f)–(6g). This process is illustrated in a three-state example in Figure 13. This completes the proof.

EC.4. Analytical results and proof for the single-station model

EC.4.1. Analytical results

ASSUMPTION 1. *The failure penalty coefficient* $\zeta \ge \Delta_t + \beta\mu_0(\mathrm{H})$*, which ensures that at least the EV driver with a 100% failure probability chooses Route C based on prior belief.*

EC.4.1.1. Public information design (PBI) Plugging the cost functions (7) into the BNE condition (3c) and noting $f_s^* \in [0,1]$, the equilibrium flow corresponding to a public signal s is given by

$$f_s^* = f_s^*(\boldsymbol{\mu}_s) = \max\left(0, \frac{\zeta - \Delta_t - \beta\mu_s(\mathrm{H})}{\alpha + \zeta}\right), \forall s \in \mathbb{S}, \tag{EC.22}$$

where $\mathbb{S} = \mathbb{W}$. Thus, $\pi(s \mid w)$ states the probability of reporting state $s \in \{\mathrm{H}, \mathrm{L}\}$ when the realized state is $w \in \{\mathrm{H}, \mathrm{L}\}$. According to Equation (2), the system's total expected cost can be represented as (see Appendix EC.4.2 for details.)

$$Z(\boldsymbol{f}) = Z(\boldsymbol{\pi}) = \sum_{w\in\mathbb{W}} \mu_0(w) \sum_{s\in\mathbb{W}} \pi(s \mid w) \left[f_s^* \left(\frac{l + \Delta_l}{v} + \alpha f_s^* + \beta\mu_s(\mathrm{H}) \right) + \left(1 - f_s^*\right) \frac{l}{v} + 0.5\zeta \left(1 - f_s^*\right)^2 \right].$$

The first term within the bracket denotes the total cost associated with charging when the public signal is s, the second term represents the total travel time incurred on Route M, and the third term is the penalty incurred due to running out of power on Route M.

No information (NI) Without any information, i.e., implementing no information policy (NI), EV drivers act on their prior belief, i.e. $\mu_s(w) = \mu_0(w), w \in \mathbb{W}, s = \emptyset$. Thus, per (EC.22), the BNE dictates the charging flow $f_\emptyset^{\mathrm{NI}*} = \frac{\zeta - \Delta_t - \beta\mu_0(\mathrm{H})}{\alpha+\zeta}$.

System optimum (SO) If the designer has a coercive power over the EV drivers, he can direct them to achieve a system optimum (SO) charging flow at each state, $f_{\mathrm{H}}^{\mathrm{SO}*}$ and $f_{\mathrm{L}}^{\mathrm{SO}*}$, without relying on persuasion. It is easy to show that, to minimize the expected system cost, the designer should set $f_{\mathrm{H}}^{\mathrm{SO}*} = \max\left(0, \frac{\zeta-\Delta_t-\beta}{2\alpha+\zeta}\right)$, $f_{\mathrm{L}}^{\mathrm{SO}*} = \frac{\zeta-\Delta_t}{2\alpha+\zeta}$. We leave it to the reader to verify that $f_\emptyset^{\mathrm{NI}*} \geq f_{\mathrm{H}}^{\mathrm{SO}*}$, which implies that the decision to withhold information from drivers leads to more EV drivers choosing charging when the station is at an abnormal state than what is deemed optimal for the system.

Full information (FI) Under a full information policy (FI), the designer always reveals the true state to all EV drivers, i.e. $\pi(s = \mathrm{H} \mid \mathrm{H}) = 1, \pi(s = \mathrm{H} \mid \mathrm{L}) = 0$. Thus, $\mu_{s=\mathrm{H}}(\mathrm{H}) = 1$ and $\mu_{s=\mathrm{L}}(\mathrm{H}) = 0$. Consequently, we have from (EC.22): $f_{\mathrm{H}}^{\mathrm{FI}*} = \max\left(0, \frac{\zeta-\Delta_t-\beta}{\alpha+\zeta}\right)$, $f_{\mathrm{L}}^{\mathrm{FI}*} = \frac{\zeta-\Delta_t}{\alpha+\zeta}$.

By comparing the solutions, we can show that $f_{\mathrm{H}}^{\mathrm{FI}*} \leq f_\emptyset^{\mathrm{NI}*} \leq f_{\mathrm{L}}^{\mathrm{FI}*}$. That is, as expected, providing full information alerts the drivers so that fewer of them would choose to charge in the abnormal state and more would do so in the normal state, compared to the situation when they receive no information. We can also show (details omitted for brevity) that $f_{\mathrm{H}}^{\mathrm{FI}*} \geq f_{\mathrm{H}}^{\mathrm{SO}*}$ and $f_{\mathrm{L}}^{\mathrm{FI}*} \geq f_{\mathrm{L}}^{\mathrm{SO}*}$, which indicates that disclosing full information may not be the best for the system. In the private information design part, we shall show a designer relying on a private information structure can hope to achieve SO in many (if not all) cases.

EC.4.1.2. Private information design (PVI) We begin by demonstrating that, under our assumptions, the ranking indicator variables $\boldsymbol{\nu}$ in Program (6) are constant — in other words, the cutoff thresholds follow a fixed order.

LEMMA 1 **(Ranking of cutoff thresholds).** *Let $\bar{r}_{\mathrm{H}}^{PVI}$ and $\bar{r}_{\mathrm{L}}^{PVI}$ be the cutoff thresholds for states* H *and* L. *We have $\bar{r}_{\mathrm{H}}^{PVI} \leq \bar{r}_{\mathrm{L}}^{PVI}$.*

Proof We prove the result by contradiction. Assume that $\bar{r}_{\mathrm{H}}^{\mathrm{PVI}} > \bar{r}_{\mathrm{L}}^{\mathrm{PVI}}$, which implies $f_{\mathrm{H}}^{\mathrm{PVI}} > f_{\mathrm{L}}^{\mathrm{PVI}}$. It follows that there must exist a driver who would receive a recommendation to skip charging in state H but to charge in state L. Let the driver's RRE be r, their obedience constraints read

$$\mu_0(\mathrm{H}) \left[J_C(f_{\mathrm{H}}^{\mathrm{PVI}} | \mathrm{H}) - J_M(p(r) | \mathrm{H}) \right] \leq 0, \qquad \mu_0(\mathrm{L}) \left[J_C(f_{\mathrm{L}}^{\mathrm{PVI}} | \mathrm{L}) - J_M(p(r) | \mathrm{L}) \right] \geq 0.$$

Since $J_M(p(r)|\mathrm{H}) = J_M(p(r)|\mathrm{L})$, the above constraints lead to $J_C(f_\mathrm{H}^{\mathrm{PVI}}|\mathrm{H}) \leq J_C(f_\mathrm{L}^{\mathrm{PVI}}|\mathrm{L})$, or (plugging the cost function) $\alpha f_\mathrm{H} + \beta \leq \alpha f_\mathrm{L}$, a contradiction with $f_\mathrm{H}^{\mathrm{PVI}} > f_\mathrm{L}^{\mathrm{PVI}}$ since $\alpha, \beta \geq 0$.

Thus, there is no need to introduce the ranking indicator $\boldsymbol{\nu}$ in this simplified setting. Accordingly, the two cutoff thresholds divide Ξ into three intervals, as identified by R_1, R_2, and R_3 in Figure 3. This allows us to write the information structure:

$$\pi(C|\mathrm{H}, r \in R_1) = 1, \quad \pi(C|\mathrm{L}, r \in R_1) = 1, \quad \pi(M|\mathrm{H}, r \in R_1) = 0, \quad \pi(M|\mathrm{L}, r \in R_1) = 0;$$

$$\pi(C|\mathrm{H}, r \in R_2) = 0, \quad \pi(C|\mathrm{L}, r \in R_2) = 1, \quad \pi(M|\mathrm{H}, r \in R_2) = 1, \quad \pi(M|\mathrm{L}, r \in R_2) = 0;$$

$$\pi(C|\mathrm{H}, r \in R_3) = 0, \quad \pi(C|\mathrm{L}, r \in R_3) = 0, \quad \pi(M|\mathrm{H}, r \in R_3) = 1, \quad \pi(M|\mathrm{L}, r \in R_3) = 1.$$

Specifically, EV drivers with $r \in R_1$ are always recommended to take Route C, and those with $r \in R_3$ are always recommended to take Route M. For drivers with $r \in R_2$, the recommendation is Route M when $w = \mathrm{H}$ and Route C when $w = \mathrm{L}$. This approach is equivalent to revealing true information for drivers in R_2, while withholding information from drivers in R_1 and R_3. Without the ranking indicator, the private information design problem can be simplified as:

$$\min\ Z\left(f_\mathrm{H}^{\mathrm{PVI}}, f_\mathrm{L}^{\mathrm{PVI}}\right) = \mu_0(\mathrm{H})\left(f_\mathrm{H}^{\mathrm{PVI}}\left(\frac{l+\Delta_l}{v} + \alpha f_\mathrm{H}^{\mathrm{PVI}} + \beta\right) + \left(1 - f_\mathrm{H}^{\mathrm{PVI}}\right)\frac{l}{v} + 0.5\zeta(1 - f_\mathrm{H}^{\mathrm{PVI}})^2\right)$$
$$+ \mu_0(\mathrm{L})\left(f_\mathrm{L}^{\mathrm{PVI}}\left(\frac{l+\Delta_l}{v} + \alpha f_\mathrm{L}^{\mathrm{PVI}}\right) + \left(1 - f_\mathrm{L}^{\mathrm{PVI}}\right)\frac{l}{v} + 0.5\zeta(1 - f_\mathrm{L}^{\mathrm{PVI}})^2\right) \tag{EC.23a}$$

subject to:

$$\mu_0(\mathrm{H})\left[\Delta_t - \zeta\left(1 - f_\mathrm{H}^{\mathrm{PVI}}\right) + \left(\alpha f_\mathrm{H}^{\mathrm{PVI}} + \beta\right)\right] + \mu_0(\mathrm{L})\left[\Delta_t - \zeta\left(1 - f_\mathrm{H}^{\mathrm{PVI}}\right) + \left(\alpha f_\mathrm{L}^{\mathrm{PVI}}\right)\right] \leq 0, \tag{EC.23b}$$

$$\mu_0(\mathrm{L})\left[\Delta_t - \zeta\left(1 - f_\mathrm{L}^{\mathrm{PVI}}\right) + \left(\alpha f_\mathrm{L}^{\mathrm{PVI}}\right)\right] \leq 0, \tag{EC.23c}$$

$$\mu_0(\mathrm{H})\left[-\Delta_t + \zeta\left(1 - f_\mathrm{H}^{\mathrm{PVI}}\right) - \left(\alpha f_\mathrm{H}^{\mathrm{PVI}} + \beta\right)\right] \leq 0, \tag{EC.23d}$$

$$\mu_0(\mathrm{H})\left[-\Delta_t + \zeta\left(1 - f_\mathrm{L}^{\mathrm{PVI}}\right) - \left(\alpha f_\mathrm{H}^{\mathrm{PVI}} + \beta\right)\right] + \mu_0(\mathrm{L})\left[-\Delta_t + \zeta\left(1 - f_\mathrm{L}^{\mathrm{PVI}}\right) - \left(\alpha f_\mathrm{L}^{\mathrm{PVI}}\right)\right] \leq 0, \tag{EC.23e}$$

$$f_\mathrm{H}^{\mathrm{PVI}}, f_\mathrm{L}^{\mathrm{PVI}} \in [0, 1]. \tag{EC.23f}$$

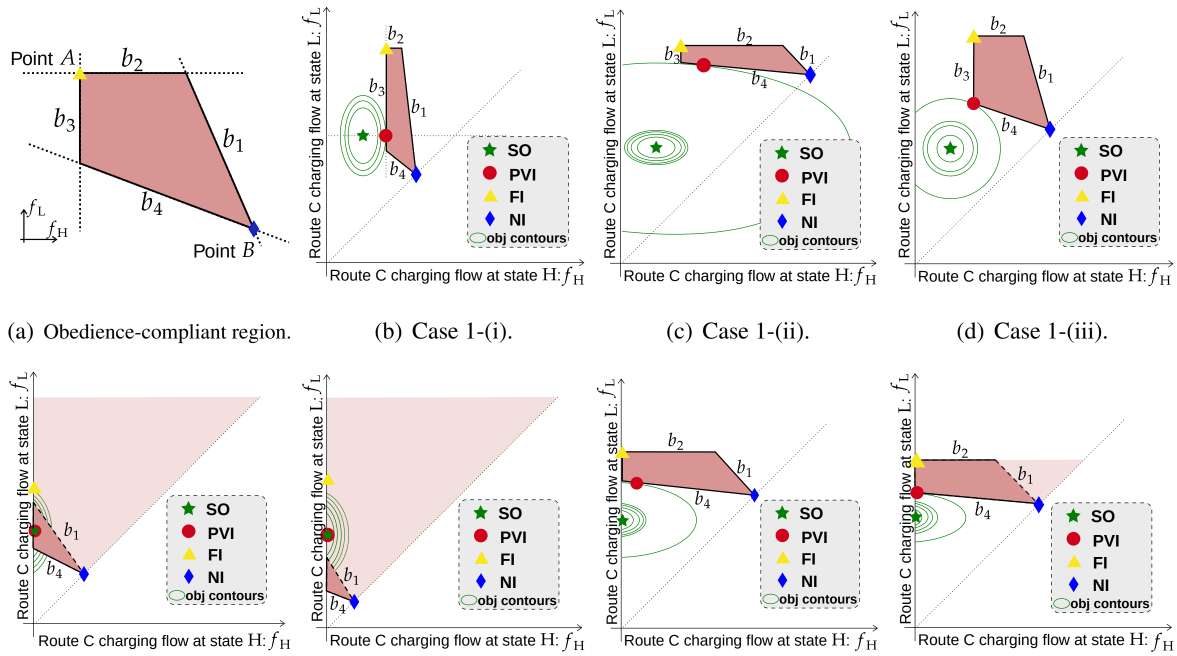
Problem (EC.23) is a convex optimization problem with linear constraints. Constraints (EC.23b) - (EC.23e) each represent a half space in $\mathbb{R}^2$. For convenience, let b_1, b_2, b_3 and b_4 denote the hyperplane bounding the half space for, respectively, Constraints (EC.23b), (EC.23c), (EC.23d), and (EC.23e). The next result characterizes the feasible set.

LEMMA 2. *The set bounded by $b_i, i = 1,2,3,4$ (see Figure 14(a)), referred to as the obedience-compliant region, is a quadrilateral in $\mathbb{R}^2$, with a right angle at the top left corner, a right or obtuse angle at the lower left and upper right corners, and a right or acute angle at the lower right corner. Moreover,*

1. *The hyperplane b_2 is a horizontal line and b_3 is a vertical line. They intersect at $\left(\frac{\zeta-\Delta_t-\beta}{\alpha+\zeta}, \frac{\zeta-\Delta_t}{\alpha+\zeta}\right)$, marked as A in Figure 14(a). When $\zeta \geq \Delta_t + \beta$, Point A corresponds to the solution under the FI policy.*

2. *The hyperplanes b_1 and b_4 both have non-positive slopes. In terms of absolute values, the slope and intercept of b_1 are greater than or equal to those of b_4. Also, b_1 and b_4 intersect at $\left(\frac{\zeta-\Delta_t-\beta\mu_0(\mathrm{H})}{\alpha+\zeta}, \frac{\zeta-\Delta_t-\beta\mu_0(\mathrm{H})}{\alpha+\zeta}\right)$, marked as B in Figure 14(a). Point B corresponds to the NI policy.*

3. *If $f_\mathrm{H}^{PVI} = 0$, then the support R_1 vanishes and hyperplane b_1 becomes redundant, as there are no travelers subject to the constraint b_1. Analogous reasoning holds for other degenerate cases, which are not realized in our setting.*



(a) Obedience-compliant region. (b) Case 1-(i). (c) Case 1-(ii). (d) Case 1-(iii).

(e) Case 2-(i). (f) Case 2-(ii). (g) Case 2-(iii). (h) Case 2-(iv).

Figure 14 **(a) Obedience-compliant region in private information design (x-axis represents f_H and y-axis represents f_L). (b)-(d): three cases of optimal solutions when $\zeta \geq \Delta_t + \beta$; (e)-(h): four possible cases of optimal solutions when $\zeta < \Delta_t + \beta$.**

Proof See Appendix EC.4.4 for details.

With Lemma 2, Problem (EC.23) can be solved analytically by enumerating all possible realizations of the feasible set. Because the solution process is tedious, we report it in Appendix EC.4.5 as Theorem 1. A key driver behind the analysis is the magnitude of ζ, the coefficient of failure on Route M, relative to the sum of Δ_t and β, the extra marginal cost on Route C when the state is abnormal (H). If $\zeta \geq \Delta_t + \beta$ — i.e., the driver with a 100% failure probability prefers Route C to Route M even when the state is always abnormal — Constraints (EC.23f) are never activated even with full information. As a result, the feasible set may manifest as one of three cases shown in Figure 14(b)-14(d).

1-(i): When $\mu_0(\mathrm{H}) \geq \frac{\alpha(\alpha+\zeta)(\zeta-\Delta_t)}{\alpha^2(\zeta-\Delta_t)+\beta\zeta(2\alpha+\zeta)}$, the optimal private design occurs on the boundary defined by b_3, giving $(f_{\mathrm{H}}^{\mathrm{FI}*}, f_{\mathrm{L}}^{\mathrm{SO}*})$ as the optimal charging flows.

1-(ii) and 1-(iii): Otherwise, the optimal private design either occurs on the boundary defined by b_4, or at the intersection of boundaries defined by b_3 and b_4.

Note that, since $f_{\mathrm{H}}^{\mathrm{FI}*} \geq f_{\mathrm{H}}^{\mathrm{SO}*}$ and $f_{\mathrm{L}}^{\mathrm{FI}*} \geq f_{\mathrm{L}}^{\mathrm{SO}*}$, i.e., the point corresponding to FI policy (the yellow triangle) must lie above and to the right of the SO point (the green star), the optimal solution can never occur on either boundary b_1 or boundary b_2.

If the failure coefficient is not large enough to meet the above condition, the obedience-compliant region will be cut by $f_{\mathrm{H}}^{\mathrm{PVI}} \geq 0$ — in other words, no one should be directed to charging in the abnormal state because of the relatively low failure coefficient. This condition leads to one of the four cases visualized in Figure 14(e)-14(h).

2-(i) and (ii): SO lies in the obedience-compliant region, thus the private information design coincides with SO solution. Note that when $f_{\mathrm{H}}^{\mathrm{PVI}} = 0$, removing b_1 expands the feasible region to include the area shaded in light pink.

2-(iii) and (iv): SO lies outside the obedience-compliant region and the optimal private design is achieved on boundary defined by b_4 with $f_{\mathrm{H}}^{\mathrm{PVI}} > 0$ and $f_{\mathrm{H}}^{\mathrm{PVI}} = 0$, respectively.

EC.4.2. Derivation for public information design

The total expected system cost, as defined in (2), is:

$$
\begin{aligned}
Z(\boldsymbol{\mu}) &= \sum_{s\in\mathbb{S}} \tau_s \mathbb{E}_{w\sim\boldsymbol{\mu}_s}\left(-v(a^*(\boldsymbol{\mu}_s), w)\right) = \sum_{s\in\mathbb{S}} \tau_s \sum_{w\in\mathbb{W}} \mu_s(w)\left(f_s^* \cdot J_C(f_s^*|w) + \int_{G^{-1}(f_s^*)}^{\bar{R}} J_M(p(r)|w) g(r)\,\mathrm{d}r\right) \\
&= \sum_{w\in\mathbb{W}} \sum_{s\in\mathbb{S}} \mu_0(w)\pi(s\mid w)\left(f_s^* \cdot J_C(f_s^*|w) + \int_{G^{-1}(f_s^*)}^{\bar{R}} J_M(p(r)|w) g(r)\,\mathrm{d}r\right),
\end{aligned}
$$

where $\tau_s = \sum_{w'\in\mathbb{W}} \mu_0(w')\pi(s\mid w')$ is the probability of receiving signal s given the information structure $\boldsymbol{\pi}$. The last equality holds by substituting the definition of the posterior belief $\mu_s(w)$, in which the denominator is τ_s. Specifically,

$$
\begin{aligned}
&Z(\boldsymbol{\pi}) = \sum_{w\in\mathbb{W}} \mu_0(w) \sum_{s\in\mathbb{W}} \pi(s\mid w)\left[f_s^* \cdot J_C(f_s^*|w) + \int_{G^{-1}(f_s^*)}^{\bar{R}} J_M(p(r)|w) g(r)\,\mathrm{d}r\right] \\
&= \sum_{w\in\mathbb{W}} \mu_0(w) \sum_{s\in\mathbb{W}} \pi(s|w)\left[f_s^* \cdot \left(\frac{l+\Delta_l}{v} + \alpha f_s^*\right) + \left(1-f_s^*\right)\frac{l}{v} + \int_{G^{-1}(f_s^*)}^{\bar{R}} \zeta\left(1-p(r)\right) g(r)\,\mathrm{d}r\right] + \sum_{s\in\mathbb{W}} \mu_0(\mathrm{H})\pi(s\mid \mathrm{H}) \cdot f_s^* \cdot \beta \\
&= \sum_{w\in\mathbb{W}} \mu_0(w) \sum_{s\in\mathbb{W}} \pi(s\mid w)\left[f_s^* \cdot \left(\frac{l+\Delta_l}{v} + \alpha f_s^* + \beta\mu_s(\mathrm{H})\right) + \left(1-f_s^*\right)\frac{l}{v} + 0.5\zeta\left(1-f_s^*\right)^2\right],
\end{aligned}
$$

where the last equation holds because

$$
\begin{aligned}
&\sum_{s\in\mathbb{W}} \mu_0(\mathrm{H})\pi(s\mid \mathrm{H}) \cdot f_s^* \cdot \beta = \sum_{s\in\mathbb{W}} \sum_{w\in\mathbb{W}} \mu_0(w)\pi(s\mid w)\frac{\mu_0(\mathrm{H})\pi(s\mid \mathrm{H})}{\sum_{w'\in\mathbb{W}} \mu_0(w')\pi(s\mid w')} \cdot f_s^* \cdot \beta \\
&= \sum_{w\in\mathbb{W}} \mu_0(w) \sum_{s\in\mathbb{W}} \pi(s\mid w)\frac{\mu_0(\mathrm{H})\pi(s\mid \mathrm{H})}{\sum_{w'\in\mathbb{W}} \mu_0(w')\pi(s\mid w')} \cdot f_s^* \cdot \beta = \sum_{w\in\mathbb{W}} \mu_0(w) \sum_{s\in\mathbb{W}} \pi(s\mid w) \cdot \mu_s(\mathrm{H}) \cdot f_s^* \cdot \beta,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{G^{-1}(f_s^*)}^{\bar{R}} \zeta\left(1-p(r)\right) g(r)\,\mathrm{d}r = \frac{\zeta}{(n-1)l}\int_{G^{-1}(f_s^*)}^{\bar{R}} (nl-r) g(r)\,\mathrm{d}r = \frac{\zeta}{(n-1)l}\left[nl(1-f_s^*) - \int_{G^{-1}(f_s^*)}^{\bar{R}} r g(r)\,\mathrm{d}r\right] \\
&= \frac{\zeta(n(1-f_s^*))}{n-1} - \frac{\zeta\int_{G^{-1}(f_s^*)}^{nl} r\,dr}{(n-1)^2 l^2} = \frac{\zeta(n(1-f_s^*))}{n-1} - \frac{\zeta(n^2l^2 - (l+(n-1)lf_s^*)^2)}{2(n-1)^2 l^2} \overset{n=2}{=} 0.5\zeta\left(1-f_s^*\right)^2.
\end{aligned}
$$

EC.4.3. Proof of Proposition 1

Our proof builds on the Bayesian persuasion framework proposed by Kamenica and Gentzkow (2011). We first briefly introduce the concavification method we used for proof. In Corollary 1 of Kamenica and Gentzkow (2011), the designer's problem is formulated as

$$
\max_{\boldsymbol{\tau}} \sum_{s\in\mathbb{S}} \tau_s \hat{v}(\boldsymbol{\mu}_s) \quad \text{subject to: } \sum_{s\in\mathbb{S}} \tau_s \mu_s(w) = \mu_0(w), \forall w\in\mathbb{W}, \tag{EC.24a}
$$

$$
\text{where } \tau_s = \sum_{w'\in\mathbb{W}} \mu_0(w')\pi(s\mid w'), s\in\mathbb{S};\ \hat{v}(\boldsymbol{\mu}_s) = \mathbb{E}_{w\sim\boldsymbol{\mu}_s}\left(v(a^*(\boldsymbol{\mu}_s), w)\right). \tag{EC.24b}
$$

Problem (EC.24) has a geometric interpretation if $|\mathbb{W}| \leq 3$. When $|\mathbb{W}| = 2$, only one of the two world states, call it w_1, is needed to specify the design objective, as visualized by the $\mu_s(w_1) - \hat{v}(\mu_s(w_1))$ plot in Figure 15. In the plot, the thick black line illustrates $\hat{v}(\mu_s(w_1))$ for a given s. Since the objective of the designer problem is the linear combination of $\hat{v}(\mu_s(w_1)), \forall s$, the decision problem is translated to maximizing that linear combination while ensuring the same linear combination of $\mu_s(w_1), \forall s$ is Bayesian plausible, i.e., the post and prior beliefs match each other.

Suppose the signal set $\mathbb{S} = \{s_1, s_2\}$, then an information structure $\boldsymbol{\pi}$ maps w_1 to $\mu_{s_i}(w_1), i = 1, 2$. The line segment connecting $\hat{v}(\mu_{s_1}(w_1))$ and $\hat{v}(\mu_{s_2}(w_1))$ (visualized as the dotted red line in Figure 15) thus represents the designer's objective function corresponding to all possible choices of τ_{s_1} and τ_{s_2} with $\tau_{s_1} + \tau_{s_2} = 1$. Bayesian plausibility implies that τ_{s_i} must be selected to reach point A, the intersection between that line segment and the dashed vertical line denoting $\mu_0(w_1)$, i.e. $\sum_{i=1}^{2} \tau_{s_i} \mu_{s_i}(w_1) = \mu_0(w_1)$. When $\boldsymbol{\pi}$ contains no useful information, $\mu_{s_i}(w_1) = \mu_0(w_1), \forall i.$, i.e., the posterior beliefs equal the prior belief. Thus, the intersection between the dashed vertical line and the thick black curve, denoted as point B, gives the designer's objective function value without information. Accordingly, the vertical distance between A and B represents the value of information for $\boldsymbol{\pi}$.

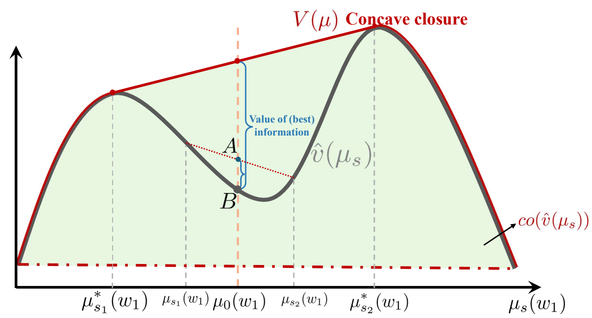


Figure 15 Illustration of the concavification method Kamenica and Gentzkow (2011).

In general, the set of all possible objective values is $\{z | (\mu_0, z) \in co(\hat{v}(\boldsymbol{\mu}_s))\}$, where $co(\hat{v}(\boldsymbol{\mu}_s))$ is the convex hull of the graph $\hat{v}(\boldsymbol{\mu}_s)$, highlighted as the green shaded area in Figure 15. Clearly, z is maximized on the *concave closure* of $\hat{v}(\mu_s(w_1))$, defined as $V(\mu_s(w_1)) \equiv \sup\{z | (\mu_s(w_1), z) \in co(\hat{v}(\boldsymbol{\mu}_s))\}$, which gives the maximum objective function value the designer can achieve for any $\mu_0(w_1)$ and $\boldsymbol{\pi}$. For a given $\mu_0(w_1)$, the optimal value of information is $\left(V(\boldsymbol{\mu}_0) - \hat{v}(\boldsymbol{\mu}_0)\right)$. Thus, Bayesian persuasion benefits the designer if and only if $V(\boldsymbol{\mu}_0) > \hat{v}(\boldsymbol{\mu}_0)$.

We next apply the analysis to our problem. From (2), the expected persuasion utility for sending s is

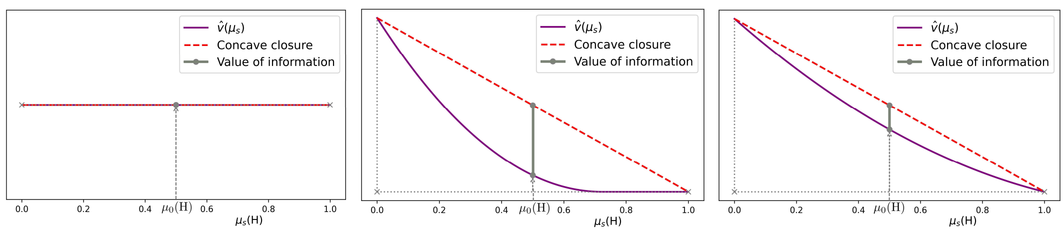
$$\hat{v}(\boldsymbol{\mu}_s) = \mathbb{E}_{w \sim \mu_s}\left(v(a^*(\boldsymbol{\mu}_s), w)\right) = -\sum_{w \in \mathbb{W}} \mu_s(w) \left[f_s^* \cdot J_C(f_s^* | w) + \int_{G^{-1}(f_s^*)}^{\bar{R}} J_M(p(r)|w) g(r)\, \mathrm{d}r \right]$$
$$= -\left(f_s^* \left(\frac{l + \Delta_l}{v} + \alpha f_s^* + \beta \mu_s(\mathrm{H}) \right) + \left(1 - f_s^*\right) \frac{l}{v} + 0.5\zeta \left(1 - f_s^*\right)^2 \right).$$

Thus, $\hat{v}(\boldsymbol{\mu}_s)$ is expressed as a function of $\mu_s(\mathrm{H})$. When $f_s^* \in (0, 1)$, the second order derivative is positive:

$$\frac{\partial^2 \hat{v}(\boldsymbol{\mu}_s)}{\partial \left(\mu_s(\mathrm{H})\right)^2} = \left[(2\alpha + \zeta) \left(\frac{-\beta}{\alpha + \zeta} \right) + \beta \right] \left(\frac{\beta}{\alpha + \zeta} \right) + \frac{\beta^2}{\alpha + \zeta} = \frac{\beta^2}{\alpha + \zeta} \left(2 - \frac{2\alpha + \zeta}{\alpha + \zeta} \right) > 0. \tag{EC.25}$$

Considering the corner solutions (see Equation (EC.22)), we analyze the following three cases separately: (i) $f_s^* = 0$ for any $\mu_s(\mathrm{H}) \in [0, 1]$; (ii) $f_s^* > 0$ when $\mu_s(\mathrm{H}) = 0$, and $f_s^* = 0$ when $\mu_s(\mathrm{H}) = 1$; (iii) $f_s^* > 0$ for any $\mu_s(\mathrm{H}) \in [0, 1]$. From (EC.22), we know that $f_s^*(\mu_s(\mathrm{H}) = 0) \geq f_s^*(\mu_s(\mathrm{H}) = 1)$, and f_s^* linearly decreases with $\mu_s(\mathrm{H})$ until reaching 0. Therefore, we can easily rule out the case where $f_s^* = 0$ when $\mu_s(\mathrm{H}) = 0$, but $f_s^* > 0$ when $\mu_s(\mathrm{H}) = 1$. Similarly, it is impossible to have $f_s^* = 1$ for all $\mu_s(\mathrm{H})$, since under our assumptions, there are always EV drivers choosing to skip charging.

In Case (i), $\hat{v}(\boldsymbol{\mu}_s)$ does not change with $\mu_s(\mathrm{H})$, the convex hull is reduced to a line, suggesting the value of information is always 0, see Figure 16(a). In Case (ii), $\hat{v}(\boldsymbol{\mu}_s)$ monotonically decreases with $\mu_s(\mathrm{H})$ until it reaches the point after which $f_s^* = 0$ and $\hat{v}(\boldsymbol{\mu}_s)$ remains constant. Until that point (i.e., $f_s^* > 0$), the function is strictly convex per (EC.25), see Figure 16(b). In Case (iii), the function $\hat{v}(\boldsymbol{\mu}_s)$ is always strictly decreasing and convex, see Figure 16(c). In Cases (ii) and (iii), thanks to the convexity of $\hat{v}(\boldsymbol{\mu}_s)$, the value of information, given by the distance between the function $\hat{v}(\boldsymbol{\mu}_s)$ (the purple solid line) and the concave closure (the red dash line for prior $\mu_0(\mathrm{H})$, is always positive. It is easy to see that, in all three cases, the concave closure is the line segment connecting $\hat{v}(\mu_{s=\mathrm{L}}(\mathrm{H}) = 0)$ and $\hat{v}(\mu_{s=\mathrm{H}}(\mathrm{H}) = 1)$. In other words, the optimal information structure $\boldsymbol{\pi}^*$ must render a posterior belief such that for $w = \mathrm{H}$, $\mu_{\mathrm{L}}(\mathrm{H}) = 0$ and $\mu_{\mathrm{H}}(\mathrm{H}) = 1$. This means $\boldsymbol{\pi}^*$ effectively equals the full revelation. This completes the proof.



(a) Case (i). (b) Case (ii). (c) Case (iii).

Figure 16 Illustration of concave closure and value of information for public information design.

EC.4.4. Proof of Lemma 2

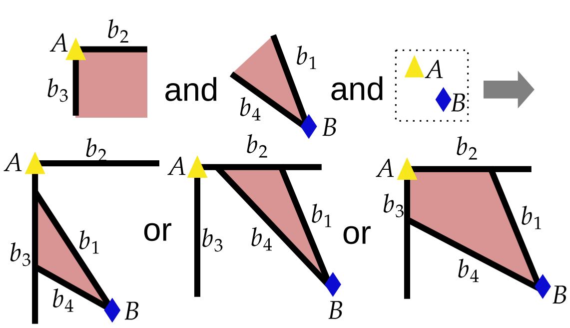


Figure 17 Geometric illustration for the proof of Lemma 2.

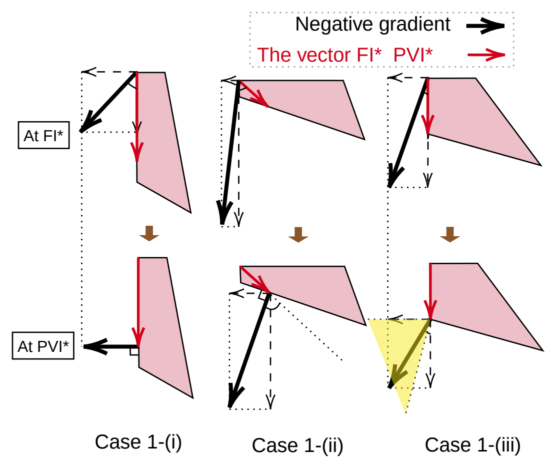


Figure 18 Illustration of the proof of Proposition 2.

Proof We first note that the slope of a hyperplane in $\mathbb{R}^2$ is the negative ratio of the coefficient of the x-coordinate (f_{H} in our case) to that of y-coordinate (f_{L} in our case) in the corresponding constraint. From (EC.23c), b_2 has a slope of 0 (hence it manifests as a horizontal line), while (EC.23d) shows b_3 must be a vertical line. Moreover, the feasible region lies below b_2 and to the right of b_3. Point A, which is where b_2 and b_3 intersect, is given by $\mu_0(\mathrm{L})\left[\Delta_t - \zeta(1 - f_{\mathrm{L}}) + (\alpha f_{\mathrm{L}})\right] = 0 \Rightarrow f_{\mathrm{L}}^A = \frac{\zeta - \Delta_t}{\alpha + \zeta}$, $\mu_0(\mathrm{H})\left[-\Delta_t + \zeta(1 - f_{\mathrm{H}}) - (\alpha f_{\mathrm{H}} + \beta)\right] = 0 \Rightarrow f_{\mathrm{H}}^A = \frac{\zeta - \Delta_t - \beta}{\alpha + \zeta}$. This proves the first statement.

For the second statement, note that b_1 can be rearranged into $(\alpha\mu_0(\mathrm{H}) + \zeta) f_{\mathrm{H}}^{\mathrm{PVI}} + \alpha\mu_0(\mathrm{L}) f_{\mathrm{L}}^{\mathrm{PVI}} + (\Delta_t - \zeta + \beta\mu_0(\mathrm{H})) = 0$, with a slope $k_1 \equiv -\frac{\alpha\mu_0(\mathrm{H}) + \zeta}{\alpha\mu_0(\mathrm{L})} \le 0$. Similarly, b_4 is arranged into $-\alpha\mu_0(\mathrm{H}) f_{\mathrm{H}}^{\mathrm{PVI}} + (\alpha\mu_0(\mathrm{L}) + \zeta) f_{\mathrm{L}}^{\mathrm{PVI}} - (\Delta_t - \zeta + \beta\mu_0(\mathrm{H})) = 0$ with a slope $k_4 \equiv -\frac{\alpha\mu_0(\mathrm{H})}{\alpha\mu_0(\mathrm{L}) + \zeta} \le 0$. The difference in the absolute value of the slopes of b_1 and b_4

is $|k_1|-|k_4| = \frac{\zeta(\alpha+\zeta)}{\alpha\mu_0(\mathrm{L})(\alpha\mu_0(\mathrm{L})+\zeta)} \geq 0$. The feasible region lies below and to the left of b_1, and above and to the right of b_4. By solving the above two equations simultaneously, we find the intersection of b_1 and b_4 as $f_{\mathrm{H}}^B = f_{\mathrm{L}}^B = \frac{\zeta-\Delta_t-\beta\mu_0(\mathrm{H})}{\alpha+\zeta}$ which is the same as the NI policy. Since $f_{\mathrm{H}}^A - f_{\mathrm{H}}^B = \frac{-\beta\mu_0(\mathrm{L})}{\alpha+\zeta} \leq 0$ and $f_{\mathrm{L}}^A - f_{\mathrm{L}}^B = \frac{\beta\mu_0(\mathrm{H})}{\alpha+\zeta} \geq 0$, we know Point A is to the upper left of Point B.

Combining the above properties implies three possible shapes for the feasible region, as enumerated in Figure 17. To ensure the shape is a quadrilateral as depicted in the lemma (rather than the triangles shown in the figure), we only need to prove that the absolute value of the slope of the line connecting points A and B, which equals $|k_{AB}| \equiv \left|\frac{f_{\mathrm{L}}^B - f_{\mathrm{L}}^A}{f_{\mathrm{H}}^B - f_{\mathrm{H}}^A}\right| = \frac{\mu_0(\mathrm{H})}{\mu_0(\mathrm{L})}$, satisfies $|k_4| \leq |k_{AB}| \leq |k_1|$. This is indeed the case because $|k_{AB}|/|k_4| = \frac{\alpha\mu_0(\mathrm{L})+\zeta}{\alpha\mu_0(\mathrm{L})} \geq 1$ and $|k_1|/|k_{AB}| = \frac{\alpha\mu_0(\mathrm{H})+\zeta}{\alpha\mu_0(\mathrm{H})} \geq 1$.

For the third statement, hyperplane b_1 is defined by Constraint (EC.23b), the obedience condition for the rightmost traveler in R_1 to follow action C. When $f_{\mathrm{H}}^{\mathrm{PVI}} = 0$, R_1 vanishes, rendering both the constraint and b_1 redundant. This completes the proof.

EC.4.5. Optimal private information design

For notational convenience, we introduce the following constants.

$$\Lambda \equiv \left(\zeta - \Delta_t - \beta\mu_0(\mathrm{H}) + \frac{\alpha\mu_0(\mathrm{H})(\Delta_t - \zeta + \beta)}{2\alpha+\zeta} + \frac{(\alpha\mu_0(\mathrm{L})+\zeta)(\Delta_t - \zeta)}{2\alpha+\zeta}\right)\left(\frac{\mu_0(\mathrm{L})(2\alpha+\zeta)}{\alpha^2\mu_0(\mathrm{H})\mu_0(\mathrm{L}) + (\alpha\mu_0(\mathrm{L})+\zeta)^2}\right) > 0,$$

$$\tilde{\mu_0} \equiv \frac{\alpha(\alpha+\zeta)(\zeta-\Delta_t)}{\alpha^2(\zeta-\Delta_t)+\beta\zeta(2\alpha+\zeta)}, \quad \underline{\mu_0} \equiv \frac{\alpha(\zeta-\Delta_t)}{-\alpha(\zeta-\Delta_t)+\beta(2\alpha+\zeta)}, \quad \overline{\mu_0} \equiv \frac{(\alpha+\zeta)(\zeta-\Delta_t)}{-\alpha(\zeta-\Delta_t)+\beta(2\alpha+\zeta)}, \quad \Omega \equiv \frac{\Lambda\alpha + (\zeta-\Delta_t-\beta)}{2\alpha+\zeta}.$$

THEOREM 1. *The optimal solution of* (EC.23) *can be described as follows.*

1. *When $\zeta \geq \Delta_t + \beta$, the optimal solution is achieved in one of the following ways:*
   - *1-(i) When $\mu_0(\mathrm{H}) \geq \tilde{\mu_0}$, we have $\left(f_{\mathrm{H}}^{PVI*}, f_{\mathrm{L}}^{PVI*}\right) = \left(\frac{\zeta-\Delta_t-\beta}{\alpha+\zeta}, \frac{\zeta-\Delta_t}{2\alpha+\zeta}\right)$, and $f_{\mathrm{H}}^{SO*} \leq f_{\mathrm{H}}^{PVI*} = f_{\mathrm{H}}^{FI*}$, and $f_{\mathrm{L}}^{PVI*} = f_{\mathrm{L}}^{SO*} \leq f_{\mathrm{L}}^{FI*}$.*
   - *1-(ii) When $\mu_0(\mathrm{H}) < \tilde{\mu_0}$ and $\Omega > f_{\mathrm{H}}^{FI*} = \frac{\zeta-\Delta_t-\beta}{\alpha+\zeta}$, we have $\left(f_{\mathrm{H}}^{PVI*}, f_{\mathrm{L}}^{PVI*}\right) = \left(\Omega, \Omega + \frac{\beta}{2\alpha+\zeta} + \frac{\Lambda\zeta}{\mu_0(\mathrm{L})(2\alpha+\zeta)}\right)$, and $f_{\mathrm{H}}^{PVI*} \geq f_{\mathrm{H}}^{SO*}$, and $f_{\mathrm{L}}^{PVI*} \geq f_{\mathrm{L}}^{SO*}$.*
   - *1-(iii) When $\mu_0(\mathrm{H}) < \tilde{\mu_0}$ and $\Omega \leq f_{\mathrm{H}}^{FI*} = \frac{\zeta-\Delta_t-\beta}{\alpha+\zeta}$, we have $\left(f_{\mathrm{H}}^{PVI*}, f_{\mathrm{L}}^{PVI*}\right) = \left(\frac{\zeta-\Delta_t-\beta}{\alpha+\zeta}, \frac{\zeta-\Delta_t-\beta\mu_0(\mathrm{H})-\alpha\mu_0(\mathrm{H})f_{\mathrm{H}}^{PVI*}}{\alpha\mu_0(\mathrm{L})+\zeta}\right)$.*
2. *When $\zeta < \Delta_t + \beta$, the feasible region is cut by $f_{\mathrm{H}}^{PVI} \geq 0$ in Constraint* (EC.23f)*. Accordingly, the optimal solution is achieved in one of the following ways.*
   - *2-(i) When $\underline{\mu_0} \leq \mu_0(\mathrm{H}) \leq \overline{\mu_0}$, we have $\left(f_{\mathrm{H}}^{PVI*}, f_{\mathrm{L}}^{PVI*}\right) = \left(f_{\mathrm{H}}^{SO*}, f_{\mathrm{L}}^{SO*}\right) = \left(0, \max\left(0, \frac{\zeta-\Delta_t}{2\alpha+\zeta}\right)\right)$.*
   - *2-(ii) When $\mu_0(\mathrm{H}) > \overline{\mu_0}$, we also have $\left(f_{\mathrm{H}}^{PVI*}, f_{\mathrm{L}}^{PVI*}\right) = \left(f_{\mathrm{H}}^{SO*}, f_{\mathrm{L}}^{SO*}\right) = \left(0, \max\left(0, \frac{\zeta-\Delta_t}{2\alpha+\zeta}\right)\right)$.*
   - *2-(iii) When $\mu_0(\mathrm{H}) < \underline{\mu_0}$ and $\Omega \geq 0$, the solution is the same as in 1-(ii).*
   - *2-(iv) When $\mu_0(\mathrm{H}) < \underline{\mu_0}$ and $\Omega < 0$, we have $\left(f_{\mathrm{H}}^{PVI*}, f_{\mathrm{L}}^{PVI*}\right) = \left(0, \frac{\zeta-\Delta_t-\beta\mu_0(\mathrm{H})}{\alpha\mu_0(\mathrm{L})+\zeta}\right)$.*

Proof Let $\lambda_i, i = 1, \ldots, 4$ be the Lagrangian multiplier for Constraints (EC.23b)–(EC.23e), respectively. Based on the shape of the compliant region and the assumption that $\zeta - \Delta_t - \beta\mu_0(\mathrm{H}) \geq 0$, point B must lie on the line segment $f_{\mathrm{H}} = f_{\mathrm{L}}$, where $f_{\mathrm{L}} \in [0,1]$. To determine whether the compliant region is within $[0,1]\times[0,1]$, it is sufficient to check whether $f_{\mathrm{H}}^A \geq 0$. This requirement leads to the condition $\zeta \geq \Delta_t + \beta$.

1. When $\zeta \geq \Delta_t + \beta$, from $f_{\mathrm{H}}^{\mathrm{FI}*} \geq f_{\mathrm{H}}^{\mathrm{SO}*}$ and $f_{\mathrm{L}}^{\mathrm{FI}*} \geq f_{\mathrm{L}}^{\mathrm{SO}*}$, we know the SO solution cannot lie inside the compliant region, and either Constraint (EC.23d), or Constraint (EC.23e), or both may be binding.

1-(i) When only Constraint (EC.23d) is binding, i.e. $\mu_0(\mathrm{H})\left[\zeta-\Delta_t-\beta-(\alpha+\zeta)\, f_{\mathrm{H}}^{\mathrm{PVI}*}\right]=0$, we have $f_{\mathrm{H}}^{\mathrm{PVI}*}=\frac{\zeta-\Delta_t-\beta}{\alpha+\zeta}$. Applying the KKT condition leads to $\mu_0(\mathrm{L})\left[\Delta_t-\zeta+(2\alpha+\zeta)\, f_{\mathrm{L}}^{\mathrm{PVI}*}\right]+\lambda_3\cdot 0=0$, which gives $f_{\mathrm{L}}^{\mathrm{PVI}*}=\frac{\zeta-\Delta_t}{2\alpha+\zeta}$. It is straightforward to verify that $f_{\mathrm{H}}^{\mathrm{PVI}*}=f_{\mathrm{H}}^{\mathrm{FI}*}$ and $f_{\mathrm{L}}^{\mathrm{PVI}*}=f_{\mathrm{L}}^{\mathrm{SO}*}$.

1-(ii) When only Constraint (EC.23e) is binding, applying the KKT condition leads to

$$\mu_0(\mathrm{H})\left(\Delta_t-\zeta+\beta+(2\alpha+\zeta)\, f_{\mathrm{H}}^{\mathrm{PVI}*}\right)+\lambda_4\left(-\alpha\mu_0(\mathrm{H})\right)=0,$$
$$\mu_0(\mathrm{L})\left(\Delta_t-\zeta+(2\alpha+\zeta)\, f_{\mathrm{L}}^{\mathrm{PVI}*}\right)+\lambda_4\left(-\alpha\mu_0(\mathrm{L})-\zeta\right)=0.$$

Adding the binding Constraint (EC.23e) and solving the resulting equation system yields $f_{\mathrm{L}}^{\mathrm{PVI}*}=\Omega+\frac{\beta}{2\alpha+\zeta}+\frac{\Lambda\zeta}{\mu_0(\mathrm{L})(2\alpha+\zeta)}$, $f_{\mathrm{H}}^{\mathrm{PVI}*}=\Omega$ where

$$\lambda_4=\left(\zeta-\Delta_t-\beta\mu_0(\mathrm{H})+\frac{\alpha\mu_0(\mathrm{H})(\Delta_t-\zeta+\beta)}{2\alpha+\zeta}+\frac{(\alpha\mu_0(\mathrm{L})+\zeta)(\Delta_t-\zeta)}{2\alpha+\zeta}\right)\left(\frac{\mu_0(\mathrm{L})(2\alpha+\zeta)}{\alpha^2\mu_0(\mathrm{H})\,\mu_0(\mathrm{L})+(\alpha\mu_0(\mathrm{L})+\zeta)^2}\right)\equiv\Lambda,$$

1-(iii) When both Constraints (EC.23d) and (EC.23e) are binding, $f_{\mathrm{H}}^{\mathrm{PVI}*}$ is the same as in 1-(i), and $f_{\mathrm{L}}^{\mathrm{PVI}*}$ is obtained by plugging $f_{\mathrm{H}}^{\mathrm{PVI}*}$ into the equation $-\alpha\mu_0(\mathrm{H})f_{\mathrm{H}}^{\mathrm{PVI}*}-(\alpha\mu_0(\mathrm{L})+\zeta)\, f_{\mathrm{L}}^{\mathrm{PVI}*}+(\zeta-\Delta_t-\beta\mu_0(\mathrm{H}))=0$.

To derive the conditions for each case, we note first that the solution obtained for Case 1-(i), $\left(f_{\mathrm{H}}^{\mathrm{FI}*}, f_{\mathrm{L}}^{\mathrm{SO}*}\right)$, is optimal if and only if it lies above b_4. The condition is necessary because if it is not satisfied, the solution would fall out of the feasible region; it is sufficient because from geometry, b_3 is the closest boundary to the optimal solution, see Figure 14(b). Requiring $\left(f_{\mathrm{H}}^{\mathrm{FI}*}, f_{\mathrm{L}}^{\mathrm{SO}*}\right)$ lies above b_4 leads to $\mu_0(\mathrm{H})\geq\frac{\alpha(\alpha+\zeta)(\zeta-\Delta_t)}{\alpha^2(\zeta-\Delta_t)+\beta\zeta(2\alpha+\zeta)}=\tilde{\mu}_0$. To obtain the additional condition that separates Case 1-(ii) from 1-(iii), we check whether the solution in Case 1-(ii) lies to the right of b_3, i.e., $\Omega>f_{\mathrm{H}}^{\mathrm{FI}*}=\frac{\zeta-\Delta_t-\beta}{\alpha+\zeta}$. If it does, then the solution does not fall on b_3, which corresponds to Case 1-(ii), see Figure 14(c); otherwise, it is Case 1-(iii), see Figure 14(d).

2. When $\zeta<\Delta_t+\beta$, the compliant region is cut by $f_{\mathrm{H}}^{\mathrm{PVI}}\geq 0$, thus $f_{\mathrm{H}}^{\mathrm{SO}*}=f_{\mathrm{H}}^{\mathrm{FI}*}=0$ in this case.

2-(i) The SO solution is feasible, it is evident that $\left(f_{\mathrm{H}}^{\mathrm{PVI}*}, f_{\mathrm{L}}^{\mathrm{PVI}*}\right)=\left(f_{\mathrm{H}}^{\mathrm{SO}*}, f_{\mathrm{L}}^{\mathrm{SO}*}\right)$, see Figure 14(e).

2-(ii) With Constraint (EC.23b) absent when $f_{\mathrm{H}}^{\mathrm{PVI}}=0$ (see Figure 14(f)), the SO solution is feasible, yielding $\left(f_{\mathrm{H}}^{\mathrm{PVI}*}, f_{\mathrm{L}}^{\mathrm{PVI}*}\right)=\left(f_{\mathrm{H}}^{\mathrm{SO}*}, f_{\mathrm{L}}^{\mathrm{SO}*}\right)$.

2-(iii) The SO solution is infeasible and only Constraint (EC.23e) is binding (see Figure 14(g)), the proof follows 1-(ii), omitted here for brevity.

2-(iv) The SO solution is infeasible, and both $f_{\mathrm{H}}^{\mathrm{PVI}*}\geq 0$ and Constraint (EC.23e) are binding (see Figure 14(h)), we have $f_{\mathrm{H}}^{\mathrm{PVI}*}=0$. Then, from $-\alpha\mu_0(\mathrm{H})\cdot 0-(\alpha\mu_0(\mathrm{L})+\zeta)\, f_{\mathrm{L}}^{\mathrm{PVI}*}+(\zeta-\Delta_t-\beta\mu_0(\mathrm{H}))=0$, we obtain $f_{\mathrm{L}}^{\mathrm{PVI}*}=\frac{\zeta-\Delta_t-\beta\mu_0(\mathrm{H})}{\alpha\mu_0(\mathrm{L})+\zeta}$.

It is worth noting that Constraint (EC.23c) can never be active at optimum, when the SO solution is infeasible. This is because activating it requires $f_{\mathrm{L}}^{\mathrm{PVI}*}=f_{\mathrm{L}}^{\mathrm{FI}*}<f_{\mathrm{L}}^{\mathrm{SO}*}$. However, we have shown $f_{\mathrm{L}}^{\mathrm{FI}*}\geq f_{\mathrm{L}}^{\mathrm{SO}*}$ in Section EC.4.1.1.

To derive the conditions for each case, we note that Case 2-(i) arises when $\left(f_{\mathrm{H}}^{\mathrm{SO}*}, f_{\mathrm{L}}^{\mathrm{SO}*}\right)$ lies below b_1 and above b_4. The former condition leads to $\mu_0(\mathrm{H})\leq\overline{\mu_0}$ and latter to $\mu_0(\mathrm{H})\geq\underline{\mu_0}$. Case 2-(ii) occurs when $\left(f_{\mathrm{H}}^{\mathrm{SO}*}, f_{\mathrm{L}}^{\mathrm{SO}*}\right)$ lies above both b_1 and b_4, which leads to $\mu_0(\mathrm{H})\geq\overline{\mu_0}$. Case 2-(iii) and (iv) occurs when $\left(f_{\mathrm{H}}^{\mathrm{SO}*}, f_{\mathrm{L}}^{\mathrm{SO}*}\right)$ lies below both b_1 and b_4, which leads to $\mu_0(\mathrm{H})\leq\underline{\mu_0}$. The difference between 2-(iii) and 2-(iv) is $\Omega=f_{\mathrm{H}}^{\mathrm{PVI}*}\geq 0$ in 2-(iii) and $\Omega<f_{\mathrm{H}}^{\mathrm{PVI}*}=0$ in 2-(iv).

This completes the proof.

EC.4.6. Proof of Proposition 2

Proof We need to show the following (the cases are defined in the proof for Theorem 1):

(a) The FI policy always outperforms the NI policy.

(b) In Cases 1-(i), 1-(ii), 1-(iii), the optimal private design (PVI) policy outperforms the FI policy.

(c) In Cases 2-(iii), 2-(iv), the optimal PVI policy outperforms the FI policy.

In Case 2-(i) and 2-(ii), the SO solution lies in the feasible region, thus the PVI policy works as well as the SO policy. In all other cases, the SO policy outperforms the PVI policy.

To prove (a), we need to show that $\delta \equiv Z\left(f_{\emptyset}^{\mathrm{NI}*}, f_{\emptyset}^{\mathrm{NI}*}\right) - Z\left(f_{\mathrm{H}}^{\mathrm{FI}*}, f_{\mathrm{L}}^{\mathrm{FI}*}\right) \geq 0$, where $Z(\cdot)$ is defined in (EC.23a). When $f_{\mathrm{H}}^{\mathrm{FI}*} > 0$, $\delta = \frac{\beta^2\zeta\mu_0(\mathrm{H})\mu_0(\mathrm{L})}{2(\alpha+\zeta)^2} \geq 0$. Otherwise, i.e., when $f_{\mathrm{H}}^{\mathrm{FI}*} = 0$, $\delta = \frac{-\zeta\mu_0(\mathrm{H})}{2(\alpha+\zeta)^2}\left[(\zeta-\Delta_t-\beta)^2 - \beta^2\mu_0(\mathrm{L})\right]$, which is non-negative as

$$\underbrace{\frac{-\zeta\mu_0(H)}{2(\alpha+\zeta)^2}}_{\leq 0}\underbrace{\left(\zeta-\Delta_t-\beta\left(1-\sqrt{\mu_0(\mathrm{L})}\right)\right)}_{\geq \zeta-\Delta_t-\beta\mu_0(\mathrm{H})\geq 0}\underbrace{\left(\zeta-\Delta_t-\beta\left(1+\sqrt{\mu_0(\mathrm{L})}\right)\right)}_{\leq \zeta-\Delta_t-\beta\leq 0} \geq 0.$$

To prove (b) and (c), we show that the designer's objective function must strictly improves as the solution moves from the FI policy $\boldsymbol{f}^{\mathrm{FI}*} = \left(f_{\mathrm{H}}^{\mathrm{FI}*}, f_{\mathrm{L}}^{\mathrm{FI}*}\right)$ to the PVI policy $\boldsymbol{f}^{\mathrm{PVI}*} = \left(f_{\mathrm{H}}^{\mathrm{PVI}*}, f_{\mathrm{L}}^{\mathrm{PVI}*}\right)$. Let $\boldsymbol{d} = \boldsymbol{f}^{\mathrm{PVI}*} - \boldsymbol{f}^{\mathrm{FI}*}$ and $\mathbb{F} = \{\boldsymbol{f} | \boldsymbol{f} = s\boldsymbol{f}^{\mathrm{FI}*} + (1-s)\boldsymbol{f}^{\mathrm{PVI}*}, \forall s \in [0,1]\}$ We proceed to prove that for any $\boldsymbol{f} \in \mathbb{F}$ we have $\langle d, -\nabla Z(\boldsymbol{f})\rangle \geq 0$.

To begin, we note that

$$-\nabla Z(\boldsymbol{f}) = \begin{bmatrix} -\frac{\partial Z(\boldsymbol{f})}{\partial f_{\mathrm{H}}^{\mathrm{PVI}}} \\ -\frac{\partial Z(\boldsymbol{f})}{\partial f_{\mathrm{L}}^{\mathrm{PVI}}} \end{bmatrix} = \begin{bmatrix} \mu_0(\mathrm{H})\left((\zeta-\Delta_t-\beta) - (2\alpha+\zeta) f_{\mathrm{H}}^{\mathrm{PVI}}\right) \\ \mu_0(\mathrm{L})\left((\zeta-\Delta_t) - (2\alpha+\zeta) f_{\mathrm{L}}^{\mathrm{PVI}}\right) \end{bmatrix}.$$

It is easy to verify that $-\frac{\partial Z(\boldsymbol{f})}{\partial f_{\mathrm{H}}^{\mathrm{PVI}}}$ decreases with $f_{\mathrm{H}}^{\mathrm{PVI}}$ and $-\frac{\partial Z(\boldsymbol{f})}{\partial f_{\mathrm{L}}^{\mathrm{PVI}}}$ decreases with $f_{\mathrm{L}}^{\mathrm{PVI}}$.

- In Case 1-(i), 1-(ii), and 1-(iii),

$$-\nabla Z(\boldsymbol{f}) = \begin{bmatrix} -\frac{\alpha\mu_0(\mathrm{H})(\zeta-\Delta_t-\beta)}{\alpha+\zeta} \\ -\frac{\alpha\mu_0(\mathrm{L})(\zeta-\Delta_t)}{\alpha+\zeta} \end{bmatrix} \leq 0$$

  at $\left(f_{\mathrm{H}}^{\mathrm{FI}*}, f_{\mathrm{L}}^{\mathrm{FI}*}\right)$, since $\zeta \geq \Delta_t + \beta$. At $\left(f_{\mathrm{H}}^{\mathrm{PVI}*}, f_{\mathrm{L}}^{\mathrm{PVI}*}\right)$,

  1-(i) we have

$$-\nabla Z(\boldsymbol{f}^{\mathrm{PVI}*}) = \begin{bmatrix} -\frac{\alpha\mu_0(\mathrm{H})(\zeta-\Delta_t-\beta)}{\alpha+\zeta} \\ 0 \end{bmatrix}.$$

  Thus, as the solution moves along $\boldsymbol{d}$ from the FI policy to the PVI policy, $-\frac{\partial Z(\boldsymbol{f})}{\partial f_{\mathrm{H}}^{\mathrm{PVI}}}$ remains a constant while $-\frac{\partial Z(\boldsymbol{f})}{\partial f_{\mathrm{L}}^{\mathrm{PVI}}}$ increases from a non-positive value to zero. Moreover, $\boldsymbol{d}$ can be written as $\tau[-1,0]^T$ for $\tau \in \mathbb{R}^+$. Thus, we have $\langle d, -\nabla Z(\boldsymbol{f})\rangle \geq 0$. From Figure 18 we can see that this means $\boldsymbol{d}$ and $-\nabla Z(\boldsymbol{f})$ always form an acute angle until $\boldsymbol{f}$ reaches $\boldsymbol{f}^{\mathrm{PVI}*}$.

  1-(ii) we have

$$-\nabla Z(\boldsymbol{f}^{\mathrm{PVI}*}) = \begin{bmatrix} -\Lambda\alpha\mu_0(\mathrm{H}) \\ -\Lambda(\alpha\mu_0(\mathrm{L})+\zeta) \end{bmatrix} \leq 0,$$

  which, per the KKT condition, must be perpendicular to b_4. Since $f_{\mathrm{H}}^{\mathrm{PVI}*} \geq f_{\mathrm{H}}^{\mathrm{FI}*}$, $f_{\mathrm{L}}^{\mathrm{PVI}*} \leq f_{\mathrm{L}}^{\mathrm{FI}*}$ (see Figure 14(c)), we can show that $-\frac{\partial Z(\boldsymbol{f})}{\partial f_{\mathrm{H}}^{\mathrm{PVI}}}$ decreases and $-\frac{\partial Z(\boldsymbol{f})}{\partial f_{\mathrm{L}}^{\mathrm{PVI}}}$ increases while the solution moves from the FI policy to the PVI policy. Consequently, the angle between $\boldsymbol{d}$ and $-\nabla Z(\boldsymbol{f})$ also increases. However, since $\boldsymbol{d}$ has a greater absolute slope than that of b_4 and $-\nabla Z(\boldsymbol{f})$ is perpendicular to b_4 at the PVI policy, the angle between the vectors at the terminal point of $\boldsymbol{d}$ cannot be obtuse, again see Figure 18. Thus, $\langle d, -\nabla Z(\boldsymbol{f})\rangle \geq 0$ for any $\boldsymbol{f} \in \mathbb{F}$.

1-(iii) it is difficult to derive the closed-form expression for $-\nabla Z(\boldsymbol{f})$. However, the result is similar to Case 1-(i): $-\frac{\partial Z(\boldsymbol{f})}{\partial f_{\mathrm{H}}^{\mathrm{PVI}}}$ remains unchanged, while $-\frac{\partial Z(\boldsymbol{f})}{\partial f_{\mathrm{L}}^{\mathrm{PVI}}}$ increases monotonically as the solution moves along $\boldsymbol{d}$. Since this means the angle between $\boldsymbol{d}$ and $-\nabla Z(\boldsymbol{f})$ will grow along $\boldsymbol{d}$, we only need to show that the angle cannot be an obtuse one at the PVI policy. Per the KKT condition, $-\nabla Z(\boldsymbol{f})$ must lie in the normal cone of the feasible set at the intersection between b_3 and b_4 (formed by the two vectors perpendicular to b_3 and b_4, see Figure 18. Its angle with $\boldsymbol{d}$, which points downward vertically and overlaps with b_3, must be acute.

- In Case 2-(iii), the proof is identical to Case 1-(ii), hence omitted for brevity. For Case 2-(iv), since $f_{\mathrm{H}}^{\mathrm{SO}*} = f_{\mathrm{H}}^{\mathrm{PVI}*} = f_{\mathrm{H}}^{\mathrm{FI}*} = 0$ and $f_{\mathrm{L}}^{\mathrm{SO}*} \leq f_{\mathrm{L}}^{\mathrm{PVI}*} \leq f_{\mathrm{L}}^{\mathrm{FI}*}$, it follows that $Z(\boldsymbol{f}^{\mathrm{PVI}*}) \leq Z(\boldsymbol{f}^{\mathrm{FI}*})$ — note that $Z(\cdot)$ is a quadratic function with respect to f_{L}, which is minimized at $f_{\mathrm{L}}^{\mathrm{SO}*}$ and have both $f_{\mathrm{L}}^{\mathrm{PVI}*}$ and $f_{\mathrm{L}}^{\mathrm{FI}*}$ lying on the same side of the minimizer $f_{\mathrm{L}}^{\mathrm{SO}*}$.

This completes the proof.

References

Bergemann D, Morris S, 2019 *Information design: A unified perspective*. *Journal of Economic Literature* 57(1):44–95.

Dughmi S, Xu H, 2016 *Algorithmic bayesian persuasion*. *Proceedings of the forty-eighth annual ACM symposium on Theory of Computing*, 412–425.

Gan J, Han M, Wu J, Xu H, 2024 *Generalized principal-agency: Contracts, information, games and beyond*. *Proceedings of the 20th Conference on Web and Internet Economics*, WINE 2024.

Kamenica E, Gentzkow M, 2011 *Bayesian persuasion*. *American Economic Review* 101(6):2590–2615.

Myerson RB, 1979 *Incentive compatibility and the bargaining problem*. *Econometrica: journal of the Econometric Society* 61–73.

Myerson RB, 1986 *Multistage games with communication*. *Econometrica: Journal of the Econometric Society* 323–358.