Appendices to “Brand Spillover as a Marketing Strategy”

Appendix A: Proofs

Proof of Lemma 1

Under sourcing structure (I, I) (i.e., both firms insource), the game becomes a straightforward Cournot duopoly game. The two firms’ optimal quantities are derived as follows: \( q_S^i = \frac{2-\theta-c}{3} \) and \( q_W^i = \frac{2\theta-1-c}{3} \). The three firms’ profits are

\[
\Pi_{CM}^i = 0, \\
\Pi_S^i = \left(\frac{2-\theta-c}{3}\right)^2, \\
\Pi_W^i = \left(\frac{2\theta-1-c}{3}\right)^2.
\]

Under (I, I), the CM obtains zero profit. Under other structures, the CM obtains a non-negative profit. Thus, the sourcing structure (I, I) is never preferred by the CM.

Under the structure (I, O) (i.e., the CM induces only Firm W to outsource while Firm S insources), for a given \( w_W^i \), the firms’ optimal quantities are \( q_S^i (w_W^i) = \frac{2-\theta-2c+w_W^i}{3} \), and \( q_W^i (w_W^i) = \frac{2\theta-1+c-2w_W^i}{3} \).

Then, the firms’ profits are

\[
\Pi_S^i (w_W^i) = \left(\frac{2-\theta-2c+w_W^i}{3}\right)^2, \\
\Pi_W^i (w_W^i) = \left(\frac{2\theta-1+c-2w_W^i}{3}\right)^2.
\]

Given the firms’ quantity responses for a given \( w_W^i \), the CM’s optimization problem is

\[
\max_{w_W^i} \Pi_{CM}^i (w_W^i) = w_W^i q_W^i, \\
\text{s.t. } \Pi_W^i (w_W^i) \geq \Pi_{CM}^i.
\]

The constraint guarantees that Firm W will accept \( w_W^i \) and outsource, which is equivalent to \( w_W^i \leq c \).

Thus, the CM’s optimal wholesale price to Firm W is \( w_W^i = \min \{ c, \frac{2\theta-1+c}{4} \} \). That is, there exists a threshold \( c_{i}^o = \frac{2\theta-1}{3} \) such that for \( c < c_{i}^o, w_W^i = c \) and \( \Pi_{CM}^i = \frac{c(2\theta-1-c)}{3} \), and for \( c \geq c_{i}^o, w_W^i = \frac{2\theta-1+c}{4} \) and \( \Pi_{CM}^i = \frac{(2\theta-1+c)^2}{24} \).

Similarly, under (O, I), there exists a threshold \( c_{i}^o = \frac{1}{3} (2-\theta) \) such that for \( c < c_{i}^o, \Pi_{CM}^i = \frac{(2-\theta-c)}{3} \), and for \( c \geq c_{i}^o, \Pi_{CM}^i = \frac{(2-\theta-c)^2}{24} \).

Now we compare the CM’s profits under (I, O) and (O, I) to determine its preference.

If \( c < c_{i}^o \), then \( \Pi_{CM}^i = \frac{(2\theta-1-c)}{3} \) and \( \Pi_C^i = \frac{(2-\theta-c)}{3} \). Thus, \( \Pi_{CM}^i - \Pi_C^i = -\frac{(1-\theta)}{3} < 0 \).

If \( c_{i}^o \leq c < c_{i}^o \), then \( \Pi_{CM}^i = \frac{(2\theta-1+c)^2}{24} \) and \( \Pi_C^i = \frac{(2\theta-1-c)^2}{24} \). Let \( \Delta_1 = \Pi_{CM}^i - \Pi_C^i \). We have \( \frac{\partial^2 \Delta_1}{\partial c^2} = \frac{3}{4} > 0 \). Thus, \( \Delta_1 \) is convex in \( c \). In addition, when \( c = c_{i}^o, \Delta_1 = -\frac{(1-\theta)(2\theta-1)}{3} < 0 \), and when \( c = c_{i}^o, \Delta_1 = -\frac{(1-\theta)(2\theta+2)}{24} < 0 \). Therefore, \( \Delta_1 < 0 \) in this case.

If \( c \geq c_{i}^o \), then \( \Pi_{CM}^i = \frac{(2\theta-1+c)^2}{24} \) and \( \Pi_C^i = \frac{(2-\theta+c)^2}{24} \). Thus, \( \Pi_{CM}^i - \Pi_C^i = -\frac{(1-\theta)(1+c+2\theta)}{24} < 0 \).

In conclusion, the CM prefers structure (O, I) to (I, O). That is, given one firm outsources, the CM is better off inducing Firm S rather than Firm W to outsource. ■
Proof of Lemma 2

Under (O, I), the CM’s profit is \( \Pi_C = w_S q_S = w_S \left( \frac{1}{2} (2 - \theta + c - 2w_S) \right) \). Clearly, \( \Pi_C \) is concave in \( w_S \). The first-order condition leads to \( w_S^1 = \frac{1}{4} (2 - \theta + c) \). In order for (O, I) to be the equilibrium structure, the optimal wholesale price \( w_S^o \leq c \) must hold so that Firm S chooses outsourcing. Therefore, the optimal price \( w_S^o = \min \{ c, \frac{1}{4} (2 - \theta + c) \} \). Solving \( c = \frac{1}{4} (2 - \theta + c) \), we have \( c = \frac{1}{3} (2 - \theta) \). That is, we have the threshold \( c^o = \frac{1}{3} (2 - \theta) \) in the lemma. ■

Proof of Lemma 3

Under (O, Ob), the CM’s profit is \( \Pi_C = w_S q_S + w_W q_W = w_S \left( \frac{1}{4} (2 - \theta - \alpha - \alpha \theta - 2w_S + w_W) \right) + w_W \left( \frac{1}{4} (2\theta + 2\alpha - 2\alpha \theta - 1 + w_S^o - 2w_W^o) \right) \). Clearly, \( \Pi_C \) is jointly concave in \( w_S \) and \( w_W \). The first-order conditions lead to \( w_S^o = \frac{1}{2} \) and \( w_W^o = \frac{1}{2} (\theta + \alpha - \alpha \theta) \).

However, the optimal wholesale prices must satisfy the firms’ participation constraint under (O, O), i.e., they choose outsourcing to the CM.

Given \( w_S \), \( w_W \), and that Firm W chooses outsourcing, Firm S’s profit by choosing outsourcing is \( \frac{1}{9} (2 - \theta - \alpha + \alpha \theta - 2w_S + w_W)^2 \), and its profit by choosing insourcing is \( \frac{1}{5} (2 - \theta - 2c + w_W)^2 \). Therefore, solving \( \frac{1}{9} (2 - \theta - \alpha + \alpha \theta - 2w_S + w_W)^2 \geq \frac{1}{5} (2 - \theta - 2c + w_W)^2 \), we derive that Firm S will choose outsourcing if and only if \( w_S \leq \frac{1}{2} (2c - \alpha + \alpha \theta) \).

Given \( w_S \), \( w_W \), and that Firm W chooses insourcing, Firm S will choose outsourcing if and only if \( w_S \leq c \).

As a result, regardless of Firm W’s sourcing strategy, Firm S will choose outsourcing if \( w_S \leq \frac{1}{2} (2c - \alpha + \alpha \theta) \).

Solving \( \frac{1}{2} = \frac{1}{2} (2c - \alpha + \alpha \theta) \), we have \( c = \frac{1}{2} (1 + \alpha - \alpha \theta) \). That is, we have the threshold \( c^o = \frac{1}{2} (1 + \alpha - \alpha \theta) \) in the lemma.

Similarly, given \( w_S \) and \( w_W \), if Firm S chooses outsourcing, Firm W will choose outsourcing if and only if \( w_W \leq c + \alpha - \alpha \theta \).

If \( c \leq c^o \), the optimal \( w_S \) is bounded. Substituting \( w_S = \frac{1}{2} (2c - \alpha + \alpha \theta) \) into the CM’s profit function and solving the first-order condition lead to the optimal \( w_W = \frac{1}{4} (2\theta + 2\alpha - 1 + \alpha - \alpha \theta) \). Solving \( \frac{1}{4} (2\theta + 2\alpha - 1 + \alpha - \alpha \theta) = c + \alpha - \alpha \theta \), we have \( c = \frac{1}{2} (2\theta - 3\alpha + 3\alpha \theta - 1) \). That is, we have the threshold \( c^o = \frac{1}{2} (2\theta - 3\alpha + 3\alpha \theta - 1) \) in the lemma.

If \( c > c^o \), the optimal \( w_S \) is not bounded, and the solution \( \frac{1}{2} (\theta + \alpha - \alpha \theta) \) is always smaller than the upper bound of the wholesale price to Firm W, \( c + \alpha - \alpha \theta \). Thus, we have the optimal \( w_W^o = \frac{1}{2} (\theta + \alpha - \alpha \theta) \) in this case.

Combining these results gives Lemma 3. ■

Proof of Proposition 1

Under (O, Ob), if \( c \leq c^o \), which can be rewritten as \( \theta \geq \frac{1 + 2c + 3\alpha}{2 + 3\alpha} \), then \( \frac{\partial^2 \Pi_C}{\partial \theta^2} = -\frac{1}{2} (5 + 2\alpha) \alpha < 0 \). Setting \( \frac{\partial \Pi_C}{\partial \theta} = 0 \) leads to \( \theta = \frac{9\alpha + 4\alpha^2 + 2c}{2\alpha(5 + 2\alpha)} \). That is, in this case \( \frac{\partial \Pi_C}{\partial \theta} \geq 0 \) if and only if \( \theta \leq \frac{9\alpha + 4\alpha^2 + 2c}{2\alpha(5 + 2\alpha)} \).

If \( c^o < c \leq c^o \), then \( \frac{\partial^2 \Pi_C}{\partial c} = -\frac{1}{2} c < 0 \), so \( \frac{\partial \Pi_C}{\partial c} \) decreases in \( c \). Setting \( \frac{\partial \Pi_C}{\partial \theta} = 0 \) leads to \( c = \frac{\alpha^2 - 8\alpha + 4\alpha^2 + 9\alpha - \alpha^2}{6\alpha} \), which is greater than \( c^o \). That is, in this case \( \frac{\partial \Pi_C}{\partial \theta} > 0 \) always holds.
If $c > c_2^{o}$, then $\frac{\partial^2 \Pi_{CM}^o}{\partial \theta^2} = \frac{1}{3} (1 - \alpha)^2 > 0$. Setting $\frac{\partial \Pi_{CM}^o}{\partial \theta} = 0$ leads to $\theta = \frac{-2c}{2(1-\alpha)}$, which is less than $\frac{\alpha}{2}$. That is, in this case, $\frac{\partial^2 \Pi_{CM}^o}{\partial \theta^2} > 0$ always holds.

To summarize, if $\theta \leq \frac{1+2\alpha+3\alpha}{2+3\alpha}$, i.e., $c \geq c_1^o$, $\Pi_{CM}^o$ always increases in $\theta$; if $\theta > \frac{1+2\alpha+3\alpha}{2+3\alpha}$, $\Pi_{CM}^o$ increases in $\theta$ if and only if $\theta \leq \frac{9\alpha+4\alpha^2+2c}{2(3+5\alpha)}$. Combining these two results leads to the proposition.

**Proof of Proposition 2**

Under $(O, Ob)$, $c \leq c_1^o$ is equivalent to $\alpha \leq \frac{2(\theta - 2c - 1)}{3(1-\theta)}$, $c_1^o < c < c_2^o$ is equivalent to $\alpha \geq \max\{\frac{2(\theta - 2c - 1)}{3(1-\theta)}; \frac{2c - 1}{1-\theta}\}$, and $c > c_2^o$ is equivalent to $\alpha < \frac{2c - 1}{1-\theta}$.

(a) If $c \leq c_1^o$, $\frac{\partial \Pi_{CM}^o}{\partial \alpha} = \frac{2}{9} (1 - \theta)(2 - \theta - c + \alpha - \alpha \theta) > 0$. If $c_1^o < c < c_2^o$, $\frac{\partial \Pi_{CM}^o}{\partial \alpha} = \frac{1}{18} (1 - \theta)(7 - 6c - 2\theta + \alpha - \alpha \theta) > 0$. If $c > c_2^o$, $\frac{\partial \Pi_{CM}^o}{\partial \alpha} = -\frac{1}{18} (1 - \theta)(2 - \alpha - \theta + \alpha \theta) < 0$. Therefore, $\Pi_{CM}^o$ increases in $\alpha$ when $c \leq c_2^o$, which is equivalent to $\alpha > \frac{2c - 1}{1-\theta}$.

(b) If $c \leq c_1^o$, $\frac{\partial \Pi_{CM}^o}{\partial \theta} = \frac{-1}{18} (1 - \theta)(4\theta - 2 - 2c + \alpha + \alpha \theta) < 0$. If $c_1^o < c < c_2^o$, $\frac{\partial \Pi_{CM}^o}{\partial \theta} = \frac{1}{18} (1 - \theta)(2\theta - 1 + 2\alpha - 2\alpha \theta) > 0$. If $c > c_2^o$, $\frac{\partial \Pi_{CM}^o}{\partial \theta} = \frac{1}{9} (1 - \theta)(2\theta - 1 - 2\alpha - 2\alpha \theta) > 0$. Therefore, $\Pi_{CM}^o$ increases in $\theta$ for $c > c_1^o$, which is equivalent to $\alpha > \frac{2(\theta - 2c - 1)}{3(1-\theta)}$.

(c) If $c \leq c_1^o$, then $\frac{\partial^2 \Pi_{CM}^o}{\partial \alpha^2} = \frac{1}{6} (1 - \theta)(5\theta + 4\alpha - 4 - 4\alpha)$, so in this case $\frac{\partial \Pi_{CM}^o}{\partial \alpha} > 0$ if and only if $\alpha < \frac{5\theta - 4}{4(1-\theta)}$. If $c_1^o < c < c_2^o$, then $\frac{\partial \Pi_{CM}^o}{\partial \alpha} = \frac{1}{12} (1 - \theta)(\alpha - \alpha \theta + 4\theta + 6\alpha - 5)$, so in this case $\frac{\partial \Pi_{CM}^o}{\partial \alpha} > 0$ if and only if $\alpha > \frac{5\theta - 4 - 2\theta + \alpha + \alpha \theta}{1-\theta}$. If $c > c_2^o$, $\frac{\partial \Pi_{CM}^o}{\partial \alpha} = \frac{1}{12} (1 - \theta)(2\alpha + 2\theta - \alpha - 1) > 0$. Combining the effect of $\alpha$ on the CM’s profit in these three cases gives that $\frac{\partial \Pi_{CM}^o}{\partial \alpha} > 0$ if and only if $(1) \alpha < \min\left\{\frac{5\theta - 4}{4(1-\theta)}, \frac{2\theta - 2c - 1}{3(1-\theta)}\right\}$; $(2) \alpha > \min\left\{\frac{5\theta - 4}{4(1-\theta)}, \frac{2\theta - 2c - 1}{3(1-\theta)}, \frac{2\alpha - 1}{1-\theta}\right\}$, or (3) $\alpha < \frac{2\alpha - 1}{1-\theta}$. Note that, if $c > \frac{1}{2}$, then we have $\frac{2\theta - 2c - 1}{3(1-\theta)} < 0$ and $\frac{5\theta - 4 - 2\theta + \alpha + \alpha \theta}{1-\theta} < 0$ ($\theta \geq \theta = \max\{\frac{2\alpha - 1}{1-\theta}, \frac{1+\alpha}{2}\}$), which has to hold so that Firm W will not be driven out of the market). Thus, the conditions for $\frac{\partial \Pi_{CM}^o}{\partial \alpha} > 0$ can be simplified as $\alpha > \frac{2\alpha - 1}{1-\theta}$ or $\alpha > \frac{2\alpha - 1}{1-\theta}$, i.e., the CM’s profit is always increasing in $\alpha$ if $c > \frac{1}{2}$. If $c < \frac{1}{2}$, then $\frac{2\alpha - 1}{1-\theta} \leq 0$; thus, the conditions for $\frac{\partial \Pi_{CM}^o}{\partial \alpha} > 0$ to hold can be simplified as $\alpha < \min\left\{\frac{5\theta - 4}{4(1-\theta)}, \frac{2\theta - 2c - 1}{3(1-\theta)}\right\}$, or $\alpha > \max\left\{\frac{5\theta - 4}{4(1-\theta)}, \frac{2\theta - 2c - 1}{3(1-\theta)}, \frac{2\alpha - 1}{1-\theta}\right\}$; moreover, if $c > \frac{8\alpha}{9}$, then we have $\frac{5\theta - 4}{4(1-\theta)} > \frac{2\theta - 2c - 1}{3(1-\theta)}$ and $\frac{5\theta - 4 - 2\theta + \alpha + \alpha \theta}{1-\theta} > 0$. Thus, if $\frac{8\alpha}{9} < c \leq \frac{1}{2}$, the conditions for $\frac{\partial \Pi_{CM}^o}{\partial \alpha} > 0$ to hold can be further simplified as $\alpha < \frac{8\alpha}{9}$, or $\alpha \geq \frac{8\alpha}{9}$; that is, the CM’s profit is always increasing in $\alpha$ if $\frac{8\alpha}{9} < c \leq \frac{1}{2}$. If $c < \min\left\{\frac{8\alpha}{9}, \frac{1}{2}\right\}$, the conditions for $\frac{\partial \Pi_{CM}^o}{\partial \alpha} > 0$ to hold can be further simplified as $\alpha < \frac{8\alpha}{9}$, or $\alpha > \frac{5\theta - 4}{4(1-\theta)}$, i.e., the CM’s profit is increasing in $\alpha$ if $\alpha$ is either sufficiently small or sufficiently large.

**Proof of Proposition 3**

In order to identify the CM’s preferred sourcing structure, we compare the CM’s profits under (O, I) and (O, Ob). Let $\Delta_{CM} = \Pi_{CM}^o - \Pi_{CM}^i$.

Scenario 1: $c < \min\{c_1^o, c_2^o\}$. We have $\frac{\partial^2 \Delta_{CM}}{\partial c^2} = -\frac{2}{3} < 0$, i.e., in this scenario, $\Delta_{CM}$ is concave in $c$. Letting $\Delta_{CM} = 0$, we have $c = \theta - \frac{1}{2} \pm \frac{1}{2} \sqrt{(2\theta - 1)^2 - 4\alpha^2 (1 + \theta)^2 - 8\alpha + 18\alpha \theta - 10\alpha^2}$. Therefore, $\Delta_{CM} > 0$ if and only if $\theta - \frac{1}{2} \pm \frac{1}{2} \sqrt{(2\theta - 1)^2 - 4\alpha^2 (1 + \theta)^2 - 8\alpha + 18\alpha \theta - 10\alpha^2} < c < \theta - \frac{1}{2} + \frac{1}{2} \sqrt{(2\theta - 1)^2 - 4\alpha^2 (1 + \theta)^2 - 8\alpha + 18\alpha \theta - 10\alpha^2}$. In addition, we have $\theta - \frac{1}{2} + \frac{1}{2} \sqrt{(2\theta - 1)^2 - 4\alpha^2 (1 + \theta)^2 - 8\alpha + 18\alpha \theta - 10\alpha^2} = \frac{1}{2} \alpha (1 - \theta) + \frac{1}{2} \sqrt{(2\theta - 1)^2 - 4\alpha^2 (1 + \theta)^2 - 8\alpha + 18\alpha \theta - 10\alpha^2} > 0$. That is, in this scenario, the condition for $\Delta_{CM} > 0$ can be written as $\theta - \frac{1}{2} - \frac{1}{2} \sqrt{(2\theta - 1)^2 - 4\alpha^2 (1 + \theta)^2 - 8\alpha + 18\alpha \theta - 10\alpha^2} < c < \min\{c_1^o, c_2^o\}$. 


Scenario 2: $c_1^{\alpha_{\Pi_1}} \leq c < c^{\alpha_{\Pi_1}}$. We have \( \frac{\partial^2 \Delta_{CM}}{\partial \alpha^2} = -\frac{1}{3} < 0 \). Letting $\Delta_{CM} = 0$, we have $c = \theta - \frac{1}{2} + \frac{3}{2} \alpha - \frac{3}{2} \alpha \theta \pm \frac{1}{2} \sqrt{100 \alpha^2 (1 - \theta)^2 + 2 - 8\theta (1 - \theta) - 16 \alpha + 36 \alpha \theta - 20 \alpha^2 \theta^2}$. Therefore, $\Delta_{CM} > 0$ if and only if $\max \left\{ c_1^{\alpha_{\Pi_1}}, \theta - \frac{1}{2} + \frac{3}{2} \alpha - \frac{3}{2} \alpha \theta - \frac{1}{2} \sqrt{100 \alpha^2 (1 - \theta)^2 + 2 - 8\theta (1 - \theta) - 16 \alpha + 36 \alpha \theta - 20 \alpha^2 \theta^2} \right\} < c < \min \left\{ c_1^{\alpha_{\Pi_1}}, \theta - \frac{1}{2} + \frac{3}{2} \alpha - \frac{3}{2} \alpha \theta + \frac{1}{2} \sqrt{100 \alpha^2 (1 - \theta)^2 + 2 - 8\theta (1 - \theta) - 16 \alpha + 36 \alpha \theta - 20 \alpha^2 \theta^2} \right\}$.

Scenario 3: $c^{\alpha_{\Pi_1}} \leq c < c_1^{\alpha_{\Pi_1}}$. We have \( \frac{\partial^2 \Delta_{CM}}{\partial \alpha^2} = -\frac{1}{3} < 0 \). Letting $\Delta_{CM} = 0$, we have $c = \frac{2}{17} + \frac{5}{17} \theta \pm \frac{2}{17} \sqrt{2 \theta^2 + 22 \theta - 16 + 153 \alpha \theta - 68 \alpha - 85 \alpha \theta^2 - 34 \alpha^2 (1 - \theta)^2}$. It is worth noting that the constraint $c^{\alpha_{\Pi_1}} \leq c < c_1^{\alpha_{\Pi_1}}$ implies $\theta > \frac{7+9\alpha}{8+9\alpha}$. Moreover, we have $c^{\alpha_{\Pi_1}} - \left( \frac{2}{17} + \frac{5}{17} \theta - \frac{2}{17} \sqrt{2 \theta^2 + 22 \theta - 16 + 153 \alpha \theta - 68 \alpha - 85 \alpha \theta^2 - 34 \alpha^2 (1 - \theta)^2} \right) = \frac{2}{17} \sqrt{2 \theta^2 + 22 \theta - 16 + 153 \alpha \theta - 68 \alpha - 85 \alpha \theta^2 - 34 \alpha^2 (1 - \theta)^2} - \left( \frac{2}{17} \right)^2$. Given $\theta > \frac{7+9\alpha}{8+9\alpha}$, we have $\frac{2}{17} \sqrt{2 \theta^2 + 22 \theta - 16 + 153 \alpha \theta - 68 \alpha - 85 \alpha \theta^2 - 34 \alpha^2 (1 - \theta)^2} > 0$. Let $\Psi = \left( \frac{2}{17} \right)^2 \left( 2 \theta^2 + 22 \theta - 16 + 153 \alpha \theta - 68 \alpha - 85 \alpha \theta^2 - 34 \alpha^2 (1 - \theta)^2 \right) - \left( \frac{2}{17} \right)^2 = \frac{32 \alpha^2 - 28 }{51}$, which is concave in $\theta$. For $\theta = \frac{7+9\alpha}{8+9\alpha}$, $\Psi = \frac{8 \alpha^2 + 20 \alpha + 55 \alpha^2}{2}$ > 0; for $\theta = 1$, $\Psi = \frac{8 \alpha^2 + 20 \alpha + 55 \alpha^2}{2}$ > 0; that is, given $\theta > \frac{7+9\alpha}{8+9\alpha}$, we have $\Psi > 0$, which is equivalent to $c^{\alpha_{\Pi_1}} - \left( \frac{2}{17} + \frac{5}{17} \theta - \frac{2}{17} \sqrt{2 \theta^2 + 22 \theta - 16 + 153 \alpha \theta - 68 \alpha - 85 \alpha \theta^2 - 34 \alpha^2 (1 - \theta)^2} \right) > 0$. Similarly, given $\theta > \frac{7+9\alpha}{8+9\alpha}$, we have $\frac{2}{17} + \frac{5}{17} \theta < \frac{2}{17} \sqrt{2 \theta^2 + 22 \theta - 16 + 153 \alpha \theta - 68 \alpha - 85 \alpha \theta^2 - 34 \alpha^2 (1 - \theta)^2} - c_1^{\alpha_{\Pi_1}} > 0$. Therefore, we always have $\Delta_{CM} > 0$ in this scenario.

Scenario 4: $c \geq \max \left\{ c^{\alpha_{\Pi_1}}, c_1^{\alpha_{\Pi_1}} \right\}$. Similar to Scenario 3, we always have $\Delta_{CM} > 0$ in this scenario. ■

**Proof of Proposition 4**

If Firm W commits to not using brand spillover, it is optimal for the CM to induce the sourcing structure (O, O). Then by Lemma 3 and setting $\alpha = 0$, Firm W’s profit is $\Pi_{W|\alpha=0}^{\alpha_{\Pi_1}} = \left( \frac{2\theta-1}{6} \right)^2$ if $c < c_{1|\alpha=0}^{\alpha_{\Pi_1}} = \frac{1}{2} (2\theta - 1)$, and $\Pi_{W|\alpha=0}^{\alpha_{\Pi_1}} = \frac{1}{2} (2\theta - 1)$ if $c_{1|\alpha=0}^{\alpha_{\Pi_1}} < c < c_{2|\alpha=0}^{\alpha_{\Pi_1}}$.

If Firm W does not make any commitment, then the equilibrium sourcing structure is derived in Proposition 3. There are two cases:

(a) If the equilibrium structure is (O, I), we know $c < c^{\alpha_{\Pi_1}}$; then by Lemma 2 we can derive $\Pi_{W}^{\alpha_{\Pi_1}} = \left( \frac{2\theta-1-c}{6} \right)^2$. Clearly, if $c \leq \frac{1}{2} (2\theta - 1)$, then $\Pi_{W}^{\alpha_{\Pi_1}} = \Pi_{W|\alpha=0}^{\alpha_{\Pi_1}}$, and committing to not using brand spillover has no impact on Firm W’s profit. But if $c > \frac{1}{2} (2\theta - 1)$, we have $\Pi_{W|\alpha=0}^{\alpha_{\Pi_1}} - \Pi_{W}^{\alpha_{\Pi_1}} = \left( \frac{2\theta-1}{6} \right)^2 - \left( \frac{2\theta-1-c}{6} \right)^2 = \frac{(6\theta-3c-2\theta+3c+2\theta+1)}{36} > 0$, and then Firm W is better off committing to not using brand spillover.

(b) If the equilibrium structure is (O, O), then we need to compare Firm W’s profit under (O, O) for a non-zero $\alpha$ (i.e., $\Pi_{W|\alpha>0}^{\alpha_{\Pi_1}}$) with its profit when committing to not using brand spillover (i.e., $\Pi_{W|\alpha=0}^{\alpha_{\Pi_1}}$). From Proposition 2, Firm W’s profit is increasing in $\alpha$ if and only if $\alpha > \frac{2\theta-2c-1}{3(1-\theta)}$, which is equivalent to $c > c_{1|\alpha=0}^{\alpha_{\Pi_1}} = \frac{1}{2} (2\theta - 3\alpha + 3\alpha \theta - 1)$. If $c > \frac{1}{2} (2\theta - 1)$, then $\Pi_{W|\alpha>0}^{\alpha_{\Pi_1}} < 0$ and $\alpha > \frac{2\theta-2c-1}{3(1-\theta)}$ always hold, and hence $\Pi_{W|\alpha>0}^{\alpha_{\Pi_1}} > \Pi_{W|\alpha=0}^{\alpha_{\Pi_1}}$ regardless of the value of $\alpha$. If $c \leq \frac{1}{2} (2\theta - 1)$ and $\alpha < 2\theta-2c-1$, then $\Pi_{W|\alpha>0}^{\alpha_{\Pi_1}} > \Pi_{W|\alpha=0}^{\alpha_{\Pi_1}}$ always holds. If $c \leq \frac{1}{2} (2\theta - 1)$ and $\alpha < \frac{2\theta-2c-1}{3(1-\theta)}$, which are equivalent to $c_{1|\alpha=0}^{\alpha_{\Pi_1}} < c < \frac{1}{2} (2\theta - 1) < c_{2|\alpha=0}^{\alpha_{\Pi_1}}$, by Lemma 3, we have $\Pi_{W|\alpha>0}^{\alpha_{\Pi_1}} = \left( \frac{2\theta-2c-2\theta+3c+2\theta+1}{6} \right)^2$ and $\Pi_{W|\alpha=0}^{\alpha_{\Pi_1}} = \left( \frac{2\theta-1-c}{6} \right)^2$. Then setting $\Pi_{W|\alpha>0}^{\alpha_{\Pi_1}} \geq \Pi_{W|\alpha=0}^{\alpha_{\Pi_1}}$ leads to $\alpha \geq \frac{2\theta-1-c}{2(1-\theta)}$. Therefore, Firm W should commit to not using brand spillover if $\alpha < \frac{2\theta-1-c}{2(1-\theta)}$. ■

**Proof of Proposition 5**
In Region 3 of Figure 2, where Firm W adopts brand spillover strategy, if \( c > c_{2}^{\infty} = \frac{1}{2} (1 + \alpha - \alpha \theta) \), \( \Pi_{S}^{\infty} = \left(\frac{\theta - \alpha \theta + 6c - 2\alpha}{6}\right)^{2} \); otherwise, \( \Pi_{S}^{\infty} = \left(\frac{\theta - \alpha \theta + 6c - 2\theta}{6}\right)^{2} \). In the absence of brand spillover, if \( c > \frac{1}{2} \), \( \Pi_{S}^{\infty} = \left(\frac{\theta - \alpha \theta}{6}\right)^{2} \); if \( \frac{1}{2} (2\theta - 1) < c \leq \frac{1}{2} \), \( \Pi_{S}^{\infty} = \left(\frac{\theta - \alpha \theta}{6}\right)^{2} \); otherwise, \( \Pi_{S}^{\infty} = \left(\frac{\theta \alpha \theta}{6}\right)^{2} \). We examine the impact of brand spillover on Firm S’s profit as follows. Let \( \Lambda_{S} \) denote the difference of Firm S’s profits with and without brand spillover.

If \( c > \frac{1}{2} (1 + \alpha - \alpha \theta) \), \( \Lambda_{S} = -\frac{1}{6} \alpha (1 - \theta) (4 - 2\theta - \alpha + \alpha \theta) < 0 \).

If \( \frac{1}{2} < c \leq \frac{1}{2} (1 + \alpha - \alpha \theta) \), \( \frac{\partial \Lambda_{S}}{\partial c} = \frac{1}{2} > 0 \). Setting \( \Lambda_{S} = 0 \) leads to \( c = \frac{3 + \alpha - \alpha \theta}{6} \) and \( c = \frac{11 - 4\theta + \alpha - \alpha \theta}{6} > \frac{1}{2} (1 + \alpha - \alpha \theta) \); thus, in this case \( \Lambda_{S} > 0 \) if \( c < \frac{3 + \alpha - \alpha \theta}{6} \); otherwise, \( \Lambda_{S} \leq 0 \).

If \( \frac{1}{2} (2\theta - 1) < c \leq \frac{1}{2} \), \( \frac{\partial \Lambda_{S}}{\partial c} = -\frac{\alpha - \alpha \theta}{12} < 0 \). Setting \( \Lambda_{S} = 0 \) leads to \( c = \frac{14 - 4\theta + \alpha - \alpha \theta}{12} > \frac{1}{2} (2\theta - 1) \); thus, in this case \( \Lambda_{S} > 0 \).

Combining these results, we find that brand spillover benefits Firm S if and only if \( c < \frac{3 + \alpha - \alpha \theta}{6} \).

Similarly, we examine the impact of brand spillover on the CM’s profit and find that brand spillover benefits the CM if and only if \( c \) is large enough. ■

**Proof of Proposition 6**

By Proposition 5, within Region 3 of Figure 4, Firm S is strictly better off with brand spillover if \( c < t_{S} \) (i.e., \( \Pi_{S}^{\infty} |_{\alpha > 0} > \Pi_{S}^{\infty} |_{\alpha = 0} \)). Along with the increase of \( \theta \), if the optimal strategy for Firm W switches from not using brand spillover (Region 1 in Figure 2) to using brand spillover (Region 3 in Figure 2), Firm S’s profit jumps from \( \Pi_{S}^{\infty} |_{\alpha = 0} \) to \( \Pi_{S}^{\infty} |_{\alpha > 0} \) across the boundary of Region 1 and Region 3. ■

**Proof of Proposition 7**

Define \( c^{\infty V} = \frac{1}{2} (2 - \theta) \),

\[
c_{1}^{\infty V} = \left( 8 (4 - \theta) (4 + \alpha (2 + \alpha) \theta) \sqrt{\theta (\alpha + (1 - \alpha) \theta)} (16 + 4\alpha - \alpha^{2} - 2\theta (2 + 3\alpha - \alpha^{2}) + \alpha (2 - \alpha) \theta^{2}) + 4\alpha \theta (1 - \theta) (\alpha + (1 - \alpha) \theta) (32 + 8\alpha + 2\alpha^{2} - (8 + 12\alpha + 5\alpha^{2}) \theta + 4\alpha (1 + \alpha) \theta^{2} - 2\alpha^{2} \theta^{3}) \right) /
\left( \theta (16 (4 - \theta) (16 - 4\theta + 2\alpha (2 - 3\theta + 2 \theta^{2}) - \alpha^{2} (1 - \theta)^{2} \sqrt{\theta (\alpha + (1 - \alpha) \theta)} - (\theta + (1 - \theta) \alpha) (\alpha^{4} \theta (1 - \theta)^{4} - 2\alpha^{3} (8 - \theta) (1 - \theta)^{3} - 8\alpha^{2} (1 - \theta)^{2} (4 - 2\theta + \theta^{2}) + 64\alpha (4 - 5\theta + \theta^{2}) + 32 (4 - \theta)^{2}) \right) 
\right),
\]

\[
c_{2}^{\infty V} = \frac{8 - 2\theta + \alpha (4 - 5\theta + \theta^{2}) - \alpha^{2} (1 - \theta)^{2}}{4 (4 - \alpha - \alpha + \alpha \theta)} \cdot T_{2}^{V} = \frac{4 \theta (4 - \theta) + 2 (8 - 8\theta - \theta^{2} + \theta^{3}) \alpha - 4 (1 - \theta)^{2} \alpha^{2} - \sqrt{\Psi}}{4 \left(2 (8 - 6\theta + \theta^{2}) - 4 (3 - 4\theta + \theta^{2}) \alpha + (1 - \theta)^{2} \alpha^{2}\right)},
\]

\[
T_{3}^{V} = \frac{4 \theta (4 - \theta) + 2 (8 - 8\theta - \theta^{2} + \theta^{3}) \alpha - 4 (1 - \theta)^{2} \alpha^{2} + \sqrt{\Psi}}{4 \left(2 (8 - 6\theta + \theta^{2}) - 4 (3 - 4\theta + \theta^{2}) \alpha + (1 - \theta)^{2} \alpha^{2}\right)},
\]

where

\[
\Psi = 2 \left(4 (4 - \theta) + 2 (2 - 3\theta + \theta^{2}) \alpha - (1 - \theta)^{2} \alpha^{2}\right) \left(4 \theta (4 - \theta) - 4 (12 - 25\theta + 16\theta^{2} - 3\theta^{3}) \alpha + (40 - 24\theta + 3\theta^{2}) (1 - \theta)^{2} \alpha^{2} - (4 - \theta) (1 - \theta)^{3} \alpha^{3}\right),
\]
Define \( T \) as the cost that satisfies \( \Pi_{CM}^{ioV} = \Pi_{CM}^{oV} \) in the interval \( c < \min \{ c_{oV}, c_{1}^{oV} \} \), which exists and is unique. The expression of \( T_{1}^{V} \) is tedious and thus omitted.

Suboptimal Sourcing Structures

In the vertical differentiation model, we first show that the CM never prefers the sourcing structures (I, I) and (I, O).

Under the sourcing structure (I, I), the two firms’ optimal quantities are \( q_{S}^{oV} = \frac{2 - \theta - c}{\theta^{2}} \) and \( q_{W}^{oV} = \frac{(1+c)^{\theta - 2} c}{\theta(4 - \theta)} \). The profits of the CM and the two firms are \( \Pi_{CM}^{oV} = 0 \), \( \Pi_{S}^{V} = \left( \frac{2 - \theta - c}{\theta^{2}} \right) \), and \( \Pi_{W}^{V} = \frac{((1+c)^{\theta - 2} c)^{2}}{\theta(4 - \theta)^{2}} \), respectively.

Under (I, I), the CM obtains zero profit. Under other structures, the CM obtains a non-negative profit.

Thus, the sourcing structure (I, I) is never preferred by the CM.

Under (I, O), the two firms’ optimal quantities are \( q_{S}^{oV} (w_{W}^{oV}) = \frac{2 - \theta - 2c + w_{W}^{oV}}{4 - \theta} \) and \( q_{W}^{oV} (w_{W}^{oV}) = \frac{(1+c)^{\theta - 2} w_{W}^{oV}}{\theta(4 - \theta)} \). Then, their profits are \( \Pi_{S}^{V} = \left( \frac{2 - \theta - 2c + w_{W}^{oV}}{4 - \theta} \right)^{2} \) and \( \Pi_{W}^{V} = \frac{((1+c)^{\theta - 2} w_{W}^{oV})^{2}}{\theta(4 - \theta)^{2}} \).

Given the two firms’ quantity responses for a given \( w_{W}^{oV} \), the CM’s optimization problem is

\[
\max_{w_{W}^{oV}} \frac{\Pi_{CM}^{oV}}{w_{W}^{oV}} = w_{W}^{oV} q_{W}^{oV} \quad \text{s.t.} \quad \Pi_{W}^{V} (w_{W}^{oV}) \geq \Pi_{CM}^{oV}.
\]

The constraint guarantees that Firm W will accept \( w_{W}^{oV} \) and outsource, and it is equivalent to \( w_{W}^{oV} \leq c \).

Thus, the CM’s optimal wholesale price to Firm W is \( w_{W}^{oV} = \min \left\{ c, \frac{(1+c)^{\theta - 2} c}{4 - \theta} \right\} \). That is, there exists a threshold \( c_{oV} = \frac{4 - \theta}{\theta^{2}} \) such that for \( c < c_{oV} \), \( w_{W}^{oV} = c \) and \( \Pi_{CM}^{oV} = \frac{((1+c)^{\theta - 2} c)^{2}}{\theta(4 - \theta)^{2}} \), and for \( c \geq c_{oV} \), \( w_{W}^{oV} = \frac{(1+c)^{\theta - 2} c}{4 - \theta} \) and \( \Pi_{CM}^{oV} = \frac{((1+c)^{\theta - 2} c)^{2}}{\theta(4 - \theta)^{2}} \).

Similarly, under (O, I), there exists a threshold \( c_{oV} = \frac{4 - \theta}{3} (2 - \theta) \) such that for \( c < c_{oV} \), \( \Pi_{CM}^{oV} = \frac{c^{2}(2 - \theta - c)}{4 - \theta} \), and for \( c \geq c_{oV} \), \( \Pi_{CM}^{oV} = \frac{(2 - \theta + c)^{2}}{\theta(4 - \theta)} \).

Now we compare the CM’s profits under (I, O) and (O, I) to identify its preference. Let \( \Delta_{I}^{V} = \Pi_{CM}^{oV} - \Pi_{CM}^{oV} \).
If $c < c^{ov}_S$, then $\Delta_Y = -\frac{c(1-\theta)(\theta+2c)}{\theta(4-\theta)} < 0$. 

If $c^{ov}_S \leq c < c^{ov}_W$, then $\frac{\partial^2 \Delta Y}{\partial c^2} = \frac{3+\theta}{4(4-\theta)} > 0$. Thus, $\Delta Y$ is convex in $c$. In addition, when $c = c^{ov}_S$, $\Delta Y = -\frac{\theta(1-\theta)(\theta+2c)}{(4-\theta)\theta} < 0$, and when $c = c^{ov}_W$, $\Delta Y = -\frac{(1-\theta)(64-25\theta+\theta^2)}{4(4-\theta)} < 0$. Therefore, $\Delta Y < 0$ in this case.

If $c \geq c^{ov}_W$, then $\Delta Y = -\frac{(1-\theta)(2+c)^2-\theta}{8(4-\theta)} < 0$.

In conclusion, the CM prefers the sourcing structure (O, I) to (I, O). That is, given that only one firm outsources, the CM is better off by inducing Firm S rather than Firm W to outsource.

**Results under (O, I)**

Next, we derive the results under the sourcing structure (O, I).

Under the sourcing structure (O, I), for a given wholesale price $w^{ov}_S$, each firm’s profit is concave in its production quantity. From the first-order conditions, we derive the quantity decisions $q^{ov}_S$ and $q^{ov}_W$.

If $\theta < \frac{1}{2}$, then $(\alpha+\theta)(1+w^{ov}_S \theta-2w^{ov}_W) \leq 2w^{ov}_S \theta+\theta(1-\alpha)\theta$. Then, the CM’s profit is $\Delta Y = \frac{(2-\theta+c-2w^{ov}_W \theta)}{4-\theta} w^{ov}_S \theta$, which is concave in $w^{ov}_S$. The first order condition leads to $w^{ov}_S = \frac{1}{2} (2-\theta+c)$.

For (O, I) to be the equilibrium sourcing structure, the optimal wholesale price $w^{ov}_S \leq c$ must hold so that Firm S chooses outsourcing to the CM. Therefore, the optimal $w^{ov}_S = \min \{c, \frac{1}{2} (2-\theta+c)\}$. Solving $c = \frac{1}{4} (2-\theta+c)$, we have $c = \frac{1}{3} (2-\theta)$. That is, there exists a threshold $c^{ov}_S = \frac{1}{3} (2-\theta)$ such that under (O, I) $w^{ov}_S = c$ if $c \leq c^{ov}_S$, and $w^{ov}_S = \frac{1}{2} (2-\theta+c)$ otherwise.

**Results under (O, O)**

Similarly, we derive the results under the sourcing structure (O, O).

Under (O, Ob), given $w^{ov}_S$ and $w^{ov}_W$, the optimal quantity responses from the first-order conditions are $q^{ov}_S = \frac{2-\theta+c-2w^{ov}_W \theta}{4-\theta+\theta(1-\alpha)\theta}$ and $q^{ov}_W = \frac{(\alpha+\theta)(1+w^{ov}_S \theta-2w^{ov}_W \theta)}{(1-\alpha)\theta(4-2\alpha+\theta(4-\alpha))}$. Then, we have that the CM’s profit is jointly concave in $w^{ov}_S$ and $w^{ov}_W$. The first-order conditions lead to $w^{ov}_S = \frac{1}{2}$ and $w^{ov}_W = \frac{1}{2} (\theta+\alpha (1-\theta))$.

However, the optimal wholesale prices must satisfy the two firms’ participation constraints under (O, Ob).

Note that, due to brand spillover, Firm W is more likely to outsource, whereas Firm S is less likely to outsource. Given $w^{ov}_S$ and $w^{ov}_W$, by comparing Firm S’s profits under different sourcing structures, we can rewrite Firm S’s participation constraint under (O, Ob) as $w^{ov}_S \leq \frac{1}{2(4-\theta)} (2c(4-\theta) - \alpha (1-\theta)(2+2c-w^{ov}_W))$.

Next, we consider three possible cases.

(i) We first consider the case with a low insourcing cost $c$ such that the optimal wholesale prices to both firms are bounded. Let $w^{ov}_S = \frac{1}{2(4-\theta)} (2c(4-\theta) - \alpha (1-\theta)(2+2c-w^{ov}_W))$. By comparing Firm W’s profits under (O, Ob) and (O, I), we derive the bounded wholesale price to Firm W

$$w^{ov}_W = \left(4(4-\theta)^2 ((4+\alpha)c - (2+(2+\alpha))(2+\alpha)) \theta \sqrt{\theta(\alpha+(1-\alpha)\theta)} + 2\theta(\alpha+(1-\alpha)\theta)(64(1+c) + 8(2+c)\alpha - 32(1+c) + 10(2+c)\alpha + (1+c)\alpha^2) \theta + (4(1+c) + 2(2+c)\alpha + 2(1+c)\alpha^2) \theta^2 - (1+c)\alpha^2 \theta^3) \right) / \left(16(4-\theta)^2 + 8\alpha(1-\theta)(4-\theta)^2 + 8\alpha^2(1-\theta)^2(2-\theta) + \alpha(1-\theta)^3 \theta, \right)$$

and similarly, the bounded wholesale price to Firm S:
\[ w_{s}^{ov} = \left( 2 (1 - \theta) (4 - \theta) ((4 + \alpha) c - (2 + (2 + \alpha) c) \theta) \alpha \sqrt{\theta (1 - \theta)} + (16 (4 - \theta)^2 - 4 (1 - c) (1 - \theta) (4 - \theta)^2 \alpha - 2 (8 + (4 - c) \theta) (1 - \theta)^2 \alpha^2 - (2c - (1 - \theta)^3 \alpha^3) \theta \right) / \left( (16 (4 - \theta)^2 + 8 \alpha (1 - \theta) \times (4 - \theta)^2 + 8 \alpha^2 (1 - \theta)^2 (2 - \theta) + \alpha^3 (1 - \theta)^3 \theta \right) \). 

(ii) We then consider the case with an intermediate \( c \) such that the optimal wholesale price to Firm W is not bounded.

Substituting \( w_{s}^{ov} = \frac{1}{2(4-\theta)} (2c (4 - \theta) - \alpha (1 - \theta) (2 + 2c - w_{w}^{ov})) \) into the CM’s profit function, we derive the optimal wholesale price to Firm W from the first-order condition

\[ w_{w}^{ov} = \left( (\theta + \alpha (1 - \theta)) (8 (1 + 2c) - 4ac - (2 (1 + 2c) - (4c - 1) \alpha) + \alpha \theta^2) \right) / \left( 8 (4 - \theta) + 4 \alpha (2 - 3 \theta + \theta^2) - 2 \alpha^2 (1 - \theta)^2 \right), \]

and similarly, the bounded wholesale price to Firm S:

\[ w_{s}^{ov} = \left( 16 (4 - \theta) c - 2 (1 - \theta) (8 + (2c - 1) \theta) \alpha - (1 - \theta)^2 (2 - \theta + 4c) \alpha^2 + (1 - \theta)^3 \alpha^3 \right) / \left( 16 (4 - \theta) + 8 \alpha (2 - 3 \theta + \theta^2) - 4a^2 (1 - \theta)^2 \right). \]

We define the threshold \( c_{1}^{ov} \) that separates the bounded and unbounded wholesale prices to Firm W as follows:

\[ c_{1}^{ov} = \left( 8 (4 - \theta) (4 + \alpha + (2 + \alpha) \theta) \sqrt{\theta (1 - \theta)} (16 + 4 \alpha - \alpha^2 - 2 \theta (2 + 3 \alpha - \alpha^2) + \alpha (2 - \alpha) \theta^2 \right) + 4 \alpha \theta (1 - \theta) (\alpha + (1 - \theta)) (32 + 8 + 2 \alpha^2 - (8 + 12 \alpha + 5 \alpha^2) \theta + 4 \alpha (1 + \alpha) \theta^2 - \alpha^2 \theta^3 \right) / \left( \theta (16 (4 - \theta) (16 - 4 \theta + 2 \alpha (2 - 3 \theta + \theta^2) - \alpha^2 (1 - \theta)^2) \sqrt{\theta (1 - \alpha) \theta - \theta + (1 - \theta) \alpha) (\alpha^4 \theta (1 - \theta)^4 - 2 \alpha^3 (8 - \theta) (1 - \theta)^3 - 8 \alpha^2 (1 - \theta)^2 (4 - 2 \theta + \theta^2) + 64 \alpha (4 - 5 \theta + \theta^2) + 32 (4 - \theta)^2) - \alpha^2 (1 - \theta)^2 \right), \]

(iii) We now consider the case with a large \( c \) such that the optimal wholesale prices to both firms are not bounded. That is, \( w_{s}^{ov} = \frac{1}{2} \) and \( w_{w}^{ov} = \frac{1}{2} (\theta + \alpha (1 - \theta)) \).

We define the threshold \( c_{2}^{ov} \) that separates the bounded and unbounded wholesale prices to Firm S as

\[ c_{2}^{ov} = \left( 8 - 2 \theta + \alpha (4 - 5 \theta + \theta^2) - \alpha^2 (1 - \theta)^2 \right) / \left( 4 (4 - \alpha + \theta) \right). \]

Finally, given the optimal decisions of all firms under (O, I) and (O, O), we can derive their corresponding profits. By comparing those profits, we can determine the CM’s preference over the sourcing structure and the impacts of brand spillover, as shown in the proposition. ■

**Proof of Proposition 8**

Define \( c^{ioP} = \frac{1}{3} (2 \theta - 1) \),

\[ T_{1}^{P} = 4 \theta - \alpha (1 - \theta) - \frac{7}{2} + \frac{1}{2} \sqrt{51 + 42 \alpha - 6 (19 + 15 \alpha) + 12 (5 + 4 \alpha) \theta^2}, \]

\[ T_{2}^{P} = \frac{6}{13} - \frac{3}{26} \theta + \frac{7}{26} \alpha (1 - \theta) - \frac{4}{26} \sqrt{(4 - \theta)^2 - 2 \alpha (8 - 23 \theta + 15 \theta^2) + 17 \alpha^2 (1 - \theta)^2}, \]
Under (I, O).

Under the structures (I, O) and (O, O).

Firm W to outsource. Finally, we identify Firm S’s preferred sourcing structure by comparing its profits

CM always prefers the structure (O, O) over (O, I).

For each outcome, we analyze the contracting between the CM and Firm W.

Contracting between the CM and Firm W

There are two possible contracting outcomes between Firm S and the CM: Firm S insources or outsources.

Let \( \Delta^P \) be the CM’s profit under (I, I) minus the CM’s profit under (O, O). Consistent with the main model, under (I, O), the CM’s optimal wholesale price to Firm W is

\[
\Delta^P_w = \frac{1}{10} + \frac{2}{3}\theta - \frac{2}{3}\alpha (1 - \theta) - \frac{1}{10} \sqrt{11 - 2\theta - 12\alpha (1 - \theta)^2 - 4\alpha^2 (1 - \theta)^2},
\]

and

\[
t^P_w = \begin{cases} 
\theta - \frac{1}{2} - \frac{1}{12} \sqrt{18 - 288\theta + 288\theta^2 - 72\alpha (1 - 3\theta + 2\theta^2) + 24\alpha^2 (1 - \theta)^2}, \text{if } c < c_{\text{opt}}, \\
1 - 2\theta + \frac{1}{4} \sqrt{9 + 72\theta - 72\theta^2 + 36\alpha (1 - 3\theta + 2\theta^2) - 12\alpha^2 (1 - \theta)^2}, \text{otherwise.}
\end{cases}
\]

It is worth noting that given the wholesale prices, the two firms’ optimal quantities in this extension are

the same as those in the main model.

For each outcome, we analyze the contracting between the CM and Firm W.

Contracting between the CM and Firm W

First, provided that Firm S insources at a unit cost of \( c \), the CM will induce Firm W to outsource because

the CM’s profit under (I, I) is zero. Consistent with the main model, under (I, O), the CM’s optimal wholesale

price to Firm W is \( w_{\text{opt}} = \min \{ c, \frac{2\theta + 1 + \alpha}{4} \} \). That is, there exists a threshold \( c_{\text{opt}} = \frac{1}{3} (2\theta - 1) \) such that

\( w_{\text{opt}} = c \) if \( c < c_{\text{opt}} \) and \( w_{\text{opt}} = \frac{2\theta + 1 + \alpha}{4} \) otherwise.

Next, provided that Firm S outsources at a wholesale price of \( w_{\text{opt}}^S \), we identify the CM’s preferred sourcing

structure by comparing the CM’s profits under the structures (O, I) and (O, O).

Under (O, I), the two firms’ quantity decisions are

\[
q_{S}^{\text{opt}} = \frac{1}{3} (2 - 2\alpha + 2w_{\text{opt}}^S + w_{\text{opt}}^W),
\]

Based on these optimal quantity responses, we can obtain the CM’s profit for a given

\( w_{\text{opt}}^S \).

Under (O, O), the two firms’ quantity decisions are

\[
q_{S}^{\text{opt}} = \frac{1}{3} (2 - \theta - \alpha + \alpha\theta - 2w_{\text{opt}}^S + w_{\text{opt}}^W) \quad \text{and} \quad q_{W}^{\text{opt}} = \frac{1}{3} (2\theta + 2\alpha - 2\alpha\theta - 1 + w_{\text{opt}}^S - 2w_{\text{opt}}^W).
\]

The CM’s optimization problem over the wholesale price is

\[
\max_{w_{\text{opt}}^W} \Pi_{\text{CM}}^{\text{opt}} = w_{\text{opt}}^S q_{S}^{\text{opt}} + w_{\text{opt}}^W q_{W}^{\text{opt}},
\]

s.t. \( \Pi_{\text{CM}}^{\text{opt}} (w_{\text{opt}}^S, w_{\text{opt}}^W) \geq \Pi_{\text{CM}}^{\text{opt}} (w_{\text{opt}}^S) \).

Solving the optimization problem, we obtain the CM’s optimal price for Firm W

\( w_{\text{opt}}^W = \min \{ c + \alpha (1 - \theta), \frac{1}{4} (2w_{S} + 2\theta - 1 + 2\alpha - 2\alpha\theta) \} \).

That is, there exists a threshold \( h_{\text{CM}}^{\text{opt}} = \frac{1}{2} + 2\alpha - \alpha (1 - \theta) \) such that

\( w_{\text{opt}}^W = c + \alpha (1 - \theta) \) if \( w_{S} > h_{\text{CM}}^{\text{opt}} \) and \( w_{\text{opt}}^W = \frac{1}{4} (2w_{S} + 2\theta - 1 + 2\alpha - 2\alpha\theta) \) otherwise. Based on the

optimal \( w_{\text{opt}}^W \) and the two firms’ quantity responses, we can obtain the CM’s profit under (O, O) for a given

\( w_{\text{opt}}^S \).

Let \( \Delta^f = \Pi_{\text{CM}}^{\text{opt}} - \Pi_{\text{CM}}^{\text{opt}} \).

If \( w_{S} > h_{\text{CM}}^{\text{opt}} \), \( \Delta^f = \frac{1}{3} (c + \alpha - \alpha\theta) (w_{S} - 1 - 2\alpha + 2\theta) \). Since \( w_{S} > h_{\text{CM}}^{\text{opt}} > 1 + 2\alpha - 2\theta \),

we have \( \Delta^f > 0 \) in this scenario. Similarly, if \( w_{S} \leq h_{\text{CM}}^{\text{opt}} \), we also have \( \Delta^f > 0 \). Thus, taking \( w_{S}^S \) as given, the

CM always prefers the structure (O, O) over (O, I).

The above analysis shows that regardless of Firm S’s outsourcing decision, the CM should always induce

Firm W to outsource. Finally, we identify Firm S’s preferred sourcing structure by comparing its profits

under the structures (I, O) and (O, O).

Firm S’s preferred sourcing structure

If Firm S chooses to insource, we can obtain Firm S’s profit based on the CM’s optimal wholesale price

under (I, O).
If Firm S chooses outsourcing, Firm S's optimization problem over the wholesale price is

$$\max_{w_{CM}^S} \Pi_{CM}^{oo} = (p_S - w_{CM}^S) q_{CM},$$

s.t. $$\Pi_{CM}^{oo} (w_{CM}^S) \geq \Pi_{CM}^{oo}.$$ 

Substituting the CM’s optimal $$w_{CM}^{oo}$$ into Firm S’s profit function, we have $$\Pi_{S}^{oo} = \frac{1}{\delta} (2 + c - \theta - 2w_{CM}^{oo})^2$$ if $$w_{CM}^{oo} > h_{CM}^{oo}$$ and $$\Pi_{S}^{oo} = \frac{1}{\delta} \left(7 - 2\alpha + 2\theta - 2\delta - 6w_{CM}^{oo}\right)^2$$ otherwise. Note that, $$w_{CM}^{oo}$$ cannot be prohibitively high to guarantee a positive production quantity. We know that Firm S’s profit is always decreasing in $$w_{CM}^{oo}$$.

Thus, Firm S will set a lowest possible wholesale price that just satisfies the CM’s participation constraint.

The CM’s profit under (I, O) is $$\Pi_{CM}^{oo} = \frac{1}{\delta} c (2\theta - 1 - c)$$ if $$c < c^{IoP}$$ and $$\Pi_{CM}^{oo} = \frac{1}{\delta} c (2\theta - 1 + c)$$ otherwise.

Therefore, we have four scenarios to examine the CM’s participation constraint.

Scenario 1: $$c < c^{IoP}$$ and $$w_{CM}^{oo} \leq h_{CM}^{oo}$$. Then the CM’s participation constraint requires $$w_{CM}^{oo} \geq \frac{1}{\delta} - \frac{1}{3} \sqrt{3 + 6c - 3\alpha + 6\alpha^2 + 3\alpha^2 - 3\theta(1 + 4c - 3\alpha + 2\alpha^2) + 3\theta^2(1 - \alpha)^2}$$. Moreover, the condition $$w_{CM}^{oo} \leq h_{CM}^{oo}$$ leads to $$c \geq \frac{1}{\delta} (\theta - \alpha + \alpha\theta)$$ or $$c \geq \frac{1}{10} + \frac{2}{5} \theta - \frac{3}{5} \alpha (1 - \theta) - \frac{1}{10} \sqrt{11 - 2\theta - 4\theta^2 - 2\alpha(11 - 27\theta + 16\theta^2) + 16\alpha^2(1 - \theta)^2}$$. We find that $$\frac{1}{\delta} \left(\theta - \alpha + \alpha\theta\right)$$ is always greater than $$\frac{1}{\delta} + \frac{2}{5} \theta - \frac{3}{5} \alpha (1 - \theta) - \frac{1}{10} \sqrt{11 - 2\theta - 4\theta^2 - 2\alpha(11 - 27\theta + 16\theta^2) + 16\alpha^2(1 - \theta)^2}$$. This, if $$c < c^{IoP}$$ and $$c \geq \frac{1}{10} + \frac{2}{5} \theta - \frac{3}{5} \alpha (1 - \theta) - \frac{1}{10} \sqrt{11 - 2\theta - 4\theta^2 - 2\alpha(11 - 27\theta + 16\theta^2) + 16\alpha^2(1 - \theta)^2}$$ is optimal for Firm S to set $$w_{CM}^{oo} = \frac{1}{\delta} \left(\theta - \alpha + \alpha\theta\right)$$.

Scenario 2: $$c < c^{IoP}$$ and $$w_{CM}^{oo} > h_{CM}^{oo}$$. Then the CM’s participation constraint requires $$w_{CM}^{oo} \geq \frac{1}{\delta} - \frac{1}{6} \sqrt{9 + 6c - 3\alpha^2 - 12\theta c - 12\alpha (1 - 3\theta + 2\theta^2) + 12\alpha^2 (1 - \theta)^2}$$. Moreover, the condition $$w_{CM}^{oo} \leq h_{CM}^{oo}$$ leads to $$c \geq \frac{1}{\delta} (\theta - \alpha + \alpha\theta)$$ or $$c \leq \frac{1}{49} \left(1 + 2\theta - 24\alpha + 24c\theta + 2\sqrt{37 + 11\theta - 61\alpha + 189\alpha\theta - 26\theta^2 + 46\theta^2 - 2\alpha(1 - \theta)(2 + 6c - 7\theta) + \alpha^2 (1 - \theta)^2}\right)$$. We find that $$\frac{1}{49} \left(1 + 2\theta - 24\alpha + 24c\theta + 2\sqrt{37 + 11\theta - 61\alpha + 189\alpha\theta - 26\theta^2 + 46\theta^2 - 2\alpha(1 - \theta)(2 + 6c - 7\theta) + \alpha^2 (1 - \theta)^2}\right)$$ is always greater than $$\frac{1}{\delta} (\theta - \alpha + \alpha\theta)$$. Thus, we have if $$c \geq c^{IoP}$$, it is optimal for Firm S to set $$w_{CM}^{oo} = \frac{1}{\delta} - \frac{1}{6} \sqrt{9 + 6c - 3\alpha^2 - 12\theta c - 12\alpha (1 - 3\theta + 2\theta^2) + 12\alpha^2 (1 - \theta)^2}$$.

Scenario 4: $$c \geq c^{IoP}$$ and $$w_{CM}^{oo} > h_{CM}^{oo}$$. Then the CM’s participation constraint requires $$w_{CM}^{oo} \geq \frac{1}{\delta} + \frac{1}{2} - \frac{1}{4} \theta - \frac{3}{4} \alpha (1 - \theta) - \frac{1}{4} \sqrt{3 + 2c + 8\theta c - 13\alpha^2 - 3\delta^2 - 2\alpha (1 - \theta)(2 + 6c - 7\theta) + \alpha^2 (1 - \theta)^2}$$. Moreover, the condition $$w_{CM}^{oo} \leq h_{CM}^{oo}$$ leads to $$c < \frac{1}{\delta} (\theta - \alpha + \alpha\theta)$$ and $$c > \frac{1}{49} \left(1 + 2\theta - 24\alpha + 24c\theta + 2\sqrt{37 + 11\theta - 61\alpha + 189\alpha\theta - 26\theta^2 + 46\theta^2 - 2\alpha(1 - \theta)(2 + 6c - 7\theta) + \alpha^2 (1 - \theta)^2}\right)$$. Since $$\frac{1}{49} \left(1 + 2\theta - 24\alpha + 24c\theta + 2\sqrt{37 + 11\theta - 61\alpha + 189\alpha\theta - 26\theta^2 + 46\theta^2 - 2\alpha(1 - \theta)(2 + 6c - 7\theta) + \alpha^2 (1 - \theta)^2}\right)$$ is always greater than $$\frac{1}{\delta} (\theta - \alpha + \alpha\theta)$$, scenario 4 can never emerge as an equilibrium.
With these optimal wholesale prices under (I, O) and (O, O), we can obtain all firms’ profits and then derive Firm S’s preferred sourcing structure and the impacts of brand spillover by comparing these profits, as shown in the proposition. ■

Appendix B: The derivation of $\theta$

As long as the optimal $q_W$ is higher than zero, Firm W will not be driven out of the market. Under the sourcing structure (O, I), substituting the CM’s optimal wholesale prices into the optimal quantity responses, we have:

If $c \leq c^i$, then $q_S^{oi} = \frac{2-\theta-c}{3}$ and $q_W^{oi} = \frac{2\theta-1-c}{3}$; clearly, $q_S^{oi}$ is always higher than zero, and $q_W^{oi} > 0$ requires $\theta > \frac{1+c}{2}$.

If $c > c^i$, then $q_S^{oi} = \frac{2-\theta+c}{6}$ and $q_W^{oi} = \frac{7\theta-2-7c}{12}$; clearly, $q_S^{oi}$ is always higher than zero, and $q_W^{oi} > 0$ requires $\theta > \frac{2+7c}{6}$.

Therefore, under (O, I), Firm W will not be driven out of the market if $\theta > \max\{\frac{1+c}{2}, \frac{2+7c}{6}\}$.

Under the sourcing structure (O, Ob), we have:

If $c \leq c_1^{oo}$, $q_S^{oo} = \frac{2-\theta-c+\alpha-\alpha\theta}{3}$ and $q_W^{oo} = \frac{4\theta-2-2\alpha-c+\alpha\theta}{6}$; clearly, $q_S^{oo}$ is always higher than zero; substituting $c = c_1^{oo}$ into $q_W^{oo}$ gives $\frac{2\theta-1+2\alpha-2\alpha\theta}{6}$, which is higher than zero if $\theta > \frac{1-2\alpha}{2(1-\alpha)}$. Since $q_W^{oo}$ decreases in $c$, $q_W^{oo} \geq \frac{2\theta-1+2\alpha-2\alpha\theta}{6}$. Furthermore, $\theta \geq \frac{\theta}{\theta} = \max\{\frac{2+7c}{6}, \frac{1+c}{2}\}$ so that Firm W is not driven out of the market, so $\theta > \frac{1+c}{2} > \frac{1-2\alpha}{2(1-\alpha)}$. Then $\frac{2\theta-1+2\alpha-2\alpha\theta}{6} > 0$ always holds and $q_W^{oo} > 0$ in this case.

If $c_1^{oo} < c \leq c_2^{oo}$, $q_S^{oo} = \frac{7-6c-2\theta+\alpha-\alpha\theta}{12}$ and $q_W^{oo} = \frac{2\theta-1+2\alpha-2\alpha\theta}{6}$; here, $q_S^{oo}$ is decreasing in $\theta$ and is higher than zero even if $\theta = 1$; $q_W^{oo} > 0$ requires $\theta > \frac{1-2\alpha}{2(1-\alpha)}$. Thus, given $\theta > \frac{1+c}{2}$, we always have $q_W^{oo} > 0$.

If $c > c_2^{oo}$, $q_S^{oo} = \frac{2-\alpha-\theta+\alpha\theta}{6}$ and $q_W^{oo} = \frac{2\theta-1+2\alpha-2\alpha\theta}{6}$; clearly, $q_S^{oo}$ is always higher than zero; given $\theta > \frac{1+c}{2}$, we always have $q_W^{oo} > 0$.

That is, under (O, O), given $\theta > \max\{\frac{1+c}{2}, \frac{2+7c}{6}\}$, no firm will be driven out of the market. Therefore, we define $\theta = \max\{\frac{1+c}{2}, \frac{2+7c}{6}\}$. ■